Journal of Statistical Physics, Vol. 82, Nos. 3/4, 1996

Surface Tension in Ising Systems with Kac Potentials

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Received January 6, 1995

We consider an Ising spin system with Kac potentials in a torus of $\mathbb{Z}^d$, $d \geqslant 2$, and fix the temperature below its Lebowitz–Penrose critical value. We prove that when the Kac scaling parameter γ vanishes, the log of the probability of an interface becomes proportional to its area and the surface tension, related to the proportionality constant, converges to the van der Waals surface tension. The results are based on the analysis of the rate functionals for Gibbsian large deviations and on the proof that they Γ-converge to the perimeter functional of geometric measure theory (which extends the notion of area). Our considerations include nonsmooth interfaces, proving that the Gibbsian probability of an interface depends only on its area and not on its regularity.

KEY WORDS: Interfaces; Kac potentials; surface tension; Γ-convergence.

1. INTRODUCTION

The thermodynamic free energy excess of a simple fluid when two pure phases coexist is

$$F = \int_\Sigma d\lambda(r)\, s_\beta(n(r)) \tag{1.1}$$

where Σ is the interface that separates the two phases, $d\lambda(r)$ is the surface area element, and $n(r)$ is a unit normal to Σ at r; finally, $s_\beta(n)$ is the surface

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0022-4715/96/0200-0743$09.50/0

tension at the inverse temperature β of a flat interface perpendicular to n. The purpose of this paper is to derive (1.1) in the context of Ising spin systems with Kac potentials. In particular we compute the surface tension, proving the validity of the expression proposed by the van der Waals theory.

In this introduction we stress the more physical aspects, leaving the mathematical details to the next sections. The surface tension is usually defined in Ising systems by a formula like

$$s_{\beta,\Lambda}(n) := \frac{1}{\beta\,|\partial\Lambda|} \log \frac{Z^{++}_{\beta,\Lambda}}{Z^{+-}_{\beta,\Lambda}} \tag{1.2}$$

where Λ is (for instance) a cube in $\mathbb{R}^d$ with the unit vector n normal to its top face, $|\partial\Lambda|$ being the corresponding area; $Z^{++}_{\beta,\Lambda}$ and $Z^{+-}_{\beta,\Lambda}$ are then the partition functions with $++$ and $+-$; respectively, boundary conditions on the top and bottom of Λ and periodic conditions on the other sides.

The relation with (1.1) comes from the assumption that the main contribution to the free energy difference when changing $+$ into $-$ in the bottom is due to the appearance of a flat interface normal on the average to n. According to (1.1), this excess free energy should then be $|\partial\Lambda|\, s_\beta(n)$, which then gives (1.2), but an exact equality can be achieved only in the thermodynamic limit where fluctuations are depressed. The existence of this thermodynamic limit has been proved for several classes of models; see, for instance, Bricmont *et al.*[2] A definition of the surface tension in terms of (1.2) is thus based on a preliminary assumption of the validity of (1.1) which conceptually should be derived first.

Let us now turn to the probabilistic aspects of the issue, which lead us to large deviations, as can be most clearly seen in a formulation where (1.1) appears again in a somewhat oblique fashion. Call m_Λ the empirical average (i.e., the total magnetization density) of the spins in the cube Λ, where the Gibbs measure with periodic boundary conditions is defined. In our context the Gibbs distribution of m_Λ when Λ invades the whole space becomes supported by two values that we call $\pm m_\beta$ (in the cases we consider there is a spin-flip symmetry). Thus $\pm m_\beta$ are the pure phase equilibrium magnetizations. The analysis of the distribution of m_Λ away from $\pm m_\beta$ is a large-deviation problem. Remarkably, in Ising systems in $d=2$ dimensions and nearest neighbor ferromagnetic interactions there is a complete answer for all temperatures below the critical one.[15,27] The probability that m_Λ is "close" to a value m in $(-m_\beta, m_\beta)$ is found to vanish as $\exp(-c\,|\partial\Lambda|)$ when $|\Lambda| \to +\infty$. The rate c comes from the solution (Wulff construction) of a variational problem involving (1.1) whose validity is an indirect consequence of the proof.

A direct proof of (1.1) is the true goal of this paper. First of all we need a well-defined setting of the problem which involves an interpretation of (1.1) as a functional in an appropriate function space. To this end it is better to regard F as a function of the magnetization profile that has Σ as its interface. We thus consider functions $u(r)$, $r \in \mathscr{T}$, with only two values $\pm m_\beta$. Then Σ is defined as the boundary of the set $\{r \in \mathscr{T}\colon u(r) = +m_\beta\}$. The minimal requirement on u for (1.1) to hold is that it should be possible to associate an area to the corresponding Σ and that this should be finite. A general notion of area has been developed in "geometric measure theory" (see, for instance, refs. 6, 17, 18, 20, and 30), where it is defined as a functional $P(u)$ with u (in our context) in $BV(\mathscr{T}; \{\pm m_\beta\})$. Then $P(u)$ generalizes the classical notion of area of the interface of u and it is finite on each element of $BV(\mathscr{T}; \{\pm m_\beta\})$. On such a space the formula (1.1) is well defined with $d\lambda(r)$ the Hausdorff measure on "the reduced boundary" of Σ,[17] provided $s_\beta(n)$ is measurable on the unit sphere of $\mathbb{R}^d$.

While smooth bounded surfaces are included, in this class there are also highly irregular surfaces still with finite area. The choice of the domain where F is defined therefore has important implications in the derivation of (1.1), namely whether it is the area the only factor that determines the probability of an interface or there are other features such as the regularity of the surface which play an effective role. We will see that for Ising systems with Kac potentials the area alone determines the probability of an interface.

The next question concerns the quantity which should play the statistical mechanical role of the functional (1.1). Since $u(r)$ is a "macroscopic density magnetization profile," the relevant quantity is the Gibbsian probability of "recognizing such a profile" out of the actual spin configurations. The region Λ where the Ising spins are defined should then be scaled down to a fixed region to be identified with the above unit torus $\mathscr{T}$ where spin configurations will be represented in terms of piecewise constant functions with values ± 1; precise definitions are given in the next section. The Gibbs measures can then be regarded as probabilities on $L^\infty(\mathscr{T}; [-1, 1])$ and we are interested in computing the probability of sets $A_\zeta(u)$ which are neighborhoods of $u \in BV(\mathscr{T}; \{\pm m_\beta\})$ (in the $L^1(\mathscr{T}; [-1, 1])$-topology, as justified in the next section) with "width" determined by ζ, a positive parameter. Calling $M_{\beta,\varepsilon,\zeta}(u)$ such a probability, where β is the inverse temperature (of the Gibbs measure) and ε the scaling parameter which gives the ratio between microscopic and macroscopic units, the quantity that approximates F in (1.1) should then be

$$\frac{\varepsilon^{d-1}}{\beta} \log M_{\beta,\varepsilon,\zeta}(u) \tag{1.3}$$

in the limit where first $\varepsilon \to 0^+$ (thermodynamic limit) and then $\zeta \to 0^+$, i.e., the limit of high accuracy in the recognition of u. The quantity ε^{d-1} scales as a surface and plays the role of $|\partial \Lambda|^{-1}$ in the previous considerations.

The quantity (1.3) has to be compared with the previous large-deviation probabilities and in fact F in (1.1) should be regarded as the rate function of large deviations associated with (1.3). It is, however, convenient at this point to particularize the discussion to the case of the Ising ferromagnetic Kac potentials[22] that we actually study in this paper. Here we have another parameter $\gamma > 0$ that determines the range of the interaction. One is interested in the limit as $\gamma \to 0^+$, where the range of the interaction diverges as γ^{-1}, as recalled in the next section. We will study the simultaneous limit where $\varepsilon \to 0^+$ (thermodynamic limit) and $\gamma \to 0^+$ (scaling limit) and for technical reasons we will ask for a strict relation between the two. One of the features of the Kac potentials is that the large-deviation rate functional as $\gamma \to 0^+$ has an explicit form in terms of the nonlocal van der Waals functional that we call $F_\varepsilon(u)$, $u \in L^\infty(\mathscr{T}; [-1, 1])$ (β is not made explicit here). According to this statement we can replace (in the limit when ε and γ vanish) (1.3) by

$$\inf_{v \in A_\zeta(u)} F_\varepsilon(v) \tag{1.4}$$

The validity of (1.1) is then in this setup just the statement that $F(u)$ in (1.1) is the limit of (1.4) as $\varepsilon \to 0^+$ and then $\zeta \to 0^+$, which on the other hand is exactly the setting of De Giorgi's notion of Γ-convergence (of F_ε to F).[4,7]

We actually prove (see Theorem 2.5) that F_ε Γ-converges to the perimeter functional $P(u)$, up to the multiplicative factor s_β, which is then the isotropic surface tension of the system. In agreement with the van der Waals theory [see (2.22)], s_β can be expressed as the free energy of the instanton solution of a nonlocal Euler–Lagrange equation for the functional F_ε.

Let us briefly describe the content of the paper. In Section 2 we state the main definitions and results; we also outline the proofs, whose details are reported in the next two sections and in the appendices. In Section 3 we show how the probability estimates involving the Gibbs measures are expressed in terms of the functional F_ε and the validity of (1.1) related to a problem of Γ-convergence, which is then solved in Section 4; more technical problems are left to the appendices. As the relation with the Γ-convergence of $\{F_\varepsilon\}$ may have an interest in its own right, we have isolated the whole argument: the problem of Γ-convergence is formulated at the end of Section 2 and studied in Section 4, which can be read independently of Section 3.

In a forthcoming paper,[1] the above results are extended to include an analysis of the Wulff and other constrained variational problems. It is also proved that the interfaces with infinite area have superexponentially small probability.

2. MAIN RESULTS

We start by recalling the notion of Ising systems with Kac potentials. While our notation may not be the most usual (see the Remarks after Definition 2.1d below), this is going to be the most convenient setup for our analysis. We split the main definition into several parts.

Definition 2.1a. *Partitions of* $\mathbb{R}^d$. For any $k \in \mathbb{Z}$, $\mathscr{D}^{(k)}$ denotes the partition of $\mathbb{R}^d$ into the d-dimensional cubes

$$\{r=(r_1,..., r_d) \in \mathbb{R}^d\colon 2^{-k}x_i \leqslant r_i < 2^{-k}(x_i+1);\ x_i \in \mathbb{Z},\ i=1,..., d\} \qquad (2.1)$$

The atoms of $\mathscr{D}^{(k)}$ are denoted by $C^{(k)}$. The atom $C^{(k)}(r)$ is the unique atom of $\mathscr{D}^{(k)}$ that contains the point r. A function $f \in L^\infty(\mathbb{R}^d)$ is $\mathscr{D}^{(k)}$-measurable if it is constant on the atoms of $\mathscr{D}^{(k)}$ and a set $A \subset \mathbb{R}^d$ is $\mathscr{D}^{(k)}$-measurable if its characteristic function $\mathbf{1}_A$ is $\mathscr{D}^{(k)}$-measurable.

Definition 2.1b. *Spin configurations.* We denote by γ a parameter that takes values in $\{2^{-k}, k \in \mathbb{N}\}$. Let $\gamma = 2^{-k_\gamma}$, $k_\gamma \in \mathbb{N}$, and $\varepsilon^{-1} \in \mathbb{N}$; we then say that σ_γ is an Ising spin configuration with mesh γ and period ε^{-1} if $\sigma_\gamma \in L^\infty(\mathbb{R}^d; \{\pm 1\})$, if σ_γ is $\mathscr{D}^{(k_\gamma)}$-measurable, and if

$$\sigma_\gamma(r) = \sigma_\gamma(r') \quad \text{whenever} \quad r_i - r'_i = \varepsilon^{-1} x_i \quad \text{where} \quad x_i \in \mathbb{Z} \quad \text{for } i = 1,..., d$$

Denoting by $\mathscr{T}_\varepsilon$ the torus in $\mathbb{R}^d$ of period ε^{-1}, we will identify a spin configuration on $\mathscr{T}_\varepsilon$ with its ε^{-1}-periodic extension to $\mathbb{R}^d$.

The values of σ_γ in each atom of $\mathscr{D}^{(k_\gamma)}$ are the spins of the configuration σ_γ.

Definition 2.1c. *Energy.* The interaction strength $J(|r|)$ is a nonnegative $\mathscr{C}^\infty$ function of $r \in \mathbb{R}^d$, supported in the unit ball, with $\sup\{s > 0\colon J(s) > 0\} = 1$ and such that

$$\int_{\mathbb{R}^d} dr\, J(|r|) = 1 \qquad (2.2)$$

Let Λ be a bounded measurable region in $\mathbb{R}^d$ and $m \in L^\infty(\Lambda; [-1, 1])$. The energy of m in Λ is defined as

$$H(m; \Lambda) := -\tfrac{1}{2} \int_\Lambda dr \int_\Lambda dr'\, J(|r-r'|)\, m(r)\, m(r') \qquad (2.3)$$

If Λ is a torus, then $|r-r'|$ in (2.3) is the distance between r and r' in the torus.

The ferromagnetic condition $J \geqslant 0$ will be essential in most of the proofs.

Definition 2.1d. *Gibbs measure.* The Gibbs measure on the torus $\mathscr{T}_\varepsilon$, with Kac potential $J(|r|)$, scaling parameter γ, and inverse temperature β, is the probability $\mu_{\beta,\gamma,\varepsilon}$ on the space of spin configurations on $\mathscr{T}_\varepsilon$ with mesh γ defined as

$$\mu_{\beta,\gamma,\varepsilon}(\sigma_\gamma) := \frac{1}{Z_{\beta,\gamma,\varepsilon}} \exp[-\beta\gamma^{-d} H(\sigma_\gamma; \mathscr{T}_\varepsilon)] \tag{2.4}$$

where

$$Z_{\beta,\gamma,\varepsilon} := \sum_{\sigma_\gamma} \exp[-\beta\gamma^{-d} H(\sigma_\gamma; \mathscr{T}_\varepsilon)] \tag{2.5}$$

is the partition function.

Remarks. Calling

$$S(x) := \sigma_\gamma(\gamma x), \qquad x \in \mathbb{Z}^d$$

and $T_\varepsilon := \{x \in \mathbb{Z}^d \colon \gamma x \in \mathscr{T}_\varepsilon\}$, we have

$$\gamma^{-d} H(\sigma_\gamma; \mathscr{T}_\varepsilon) = -\tfrac{1}{2} \sum_{\substack{x,\, y \in T_\varepsilon \\ x \neq y}} J_\gamma(x, y)\, S(x)\, S(y) + c_\varepsilon \tag{2.6}$$

where c_ε (which takes into account the sum over $x = y$) is independent of σ_γ, and, recalling $\gamma = 2^{-k_\gamma}$,

$$J_\gamma(x, y) := \gamma^{-d} \int_{C^{(k_\gamma)}(\gamma x)} dr \int_{C^{(k_\gamma)}(\gamma y)} dr'\, J(|r-r'|) \tag{2.7}$$

Observe that the coefficient c_ε drops from the expression for the Gibbs measure and it is therefore irrelevant. Then, using the variables $S(x)$, we find that the energy and the Gibbs formula take the usual form; in particular, neglecting the variation of J in the integral in (2.7), we get

$$J_\gamma(x, y) \approx \gamma^d J(\gamma\, |x-y|) \tag{2.8}$$

which has the typical scaling properties of the original Kac potential. To simplify the notation we have directly defined the model with the interaction (2.7), but the results in this paper hold as well when the energy is given by (2.6) with (2.8) holding as an equality.

The system in Definition 2.1 is thus included in the class introduced by Kac to model the van der Waals theory of phase transition, which is in fact derived by taking the limit $\gamma \to 0^+$.[22] The physically correct procedure would be to first let $\varepsilon^{-1} \to +\infty$ and then $\gamma \to 0^+$, but we will instead study the much simpler problem where ε^{-1} diverges "not too fast" as $\gamma \to 0^+$; see (2.10) below. As we shall see, even in this regime there are interesting phenomena.

Definition 2.1e. *Choice of parameters.* In the sequel we fix $\beta > 1$ and, setting

$$0 < \alpha < \frac{1}{d+1} \tag{2.9}$$

we choose

$$\varepsilon^{-1} := [\gamma^{-\alpha}] \tag{2.10}$$

where $[a]$ denotes the integer part of a.

The system with $\beta > 1$ has a phase transition when $\gamma \to 0^+$, that is, in the Lebowitz–Penrose limit.[23] There are two equilibrium magnetizations, $\pm m_\beta$, where

$$m_\beta = \tanh\{\beta m_\beta\}, \qquad m_\beta > 0 \tag{2.11}$$

(which has a solution $0 < m_\beta < 1$ if and only if $\beta > 1$). The equilibrium magnetizations are defined in ref. 23 in terms of the partition function. We will extend the result by showing that $\pm m_\beta$ are also the magnetization densities of the typical Gibbs spin configurations. We will in fact prove in Theorem 2.3 that the probability of the configurations which have either magnetization m_β or $-m_\beta$ converges in the limit $\gamma \to 0^+$ to 1. We will then investigate the residual configurations, in particular those which have an interface. Since by Theorem 2.3 they have vanishing probability, this will be a problem of large deviations, but, as we shall see [Eq. (2.16)], with an "anomalous normalization."

For any $r \in \mathbb{R}^d$ and $R > 0$ we set $B_R(r) = \{r' \in \mathbb{R}^d\colon |r - r'| \leqslant R\}$ and use the shorthand notation

$$\fint_{B_R(r)} dr'\, f(r') = \frac{1}{|B_R(r)|} \int_{B_R(r)} dr'\, f(r') \tag{2.12}$$

Definition 2.2. Given $-1 \leqslant m \leqslant 1$, $R > 0$, and $\zeta > 0$, we say that a spin configuration σ_γ on $\mathscr{T}_\varepsilon$ has magnetization constantly equal to m with accuracy (R, ζ) if

$$\varepsilon^d \int_{\mathscr{T}_\varepsilon} dr \left| m - \fint_{B_R(r)} dr' \, \sigma_\gamma(r') \right| < \zeta \tag{2.13}$$

and write $\sigma_\gamma \in \mathscr{P}^{\pm}_{R,\zeta,\gamma}$ if σ_γ satisfies (2.13) with $m = \pm m_\beta$.

It is clearly necessary to define the magnetization of a spin configuration via an averaging procedure, because the spins have only values ± 1. There is, however, some degree of arbitrariness about the size of the averaging region; we have chosen regions with finite volumes (in interaction range units). Observe that (2.13) does not imply that the averages are uniformly close to m, but that this only happens in a large fraction of the whole volume. Such a weaker condition is more likely to extend to systems where the condition that α in (2.9) is small is either relaxed or dropped.

In the next section we will prove the following result.

Theorem 2.3. Let α, ε be as in (2.9), (2.10), respectively, and let $R > 0$. Then there is $\zeta_\gamma \to 0^+$ as $\gamma \to 0^+$ so that

$$\lim_{\gamma \to 0^+} \mu_{\beta,\gamma,\varepsilon}(\mathscr{P}^-_{R,\zeta_\gamma,\gamma} \cup \mathscr{P}^+_{R,\zeta_\gamma,\gamma}) = 1 \tag{2.14}$$

In the complement of $\mathscr{P}^-_{R,\zeta_\gamma,\gamma} \cup \mathscr{P}^+_{R,\zeta_\gamma,\gamma}$ there are configurations that describe coexistence of phases and interfaces; to see this we need to generalize Definition 2.2, replacing the number m by a function $m(\cdot)$. Denoting by $\mathscr{T}$ the unit torus in $\mathbb{R}^d$ and by $L^\infty(\mathscr{T}; [-1, 1])$ the space of integrable functions on $\mathscr{T}$ with values in $[-1, 1]$, we set:

Definition 2.4. Let $m \in L^\infty(\mathscr{T}; [-1, 1])$ and $R, \zeta \in \mathbb{R}^+$. We say that a spin configuration σ_γ on $\mathscr{T}_\varepsilon$ has magnetization m with accuracy (R, ζ) if $\sigma_\gamma \in \mathscr{P}_{R,\zeta,\gamma}(m)$, where

$$\mathscr{P}_{R,\zeta,\gamma}(m) = \Big\{ m^* \in L^\infty(\mathscr{T}_\varepsilon; [-1, 1]): \\ \varepsilon^d \int_{\mathscr{T}_\varepsilon} dr \left| \fint_{B_R(r)} dr' \, (m(\varepsilon r') - m^*(r')) \right| < \zeta \Big\} \tag{2.15}$$

Let $m \in L^\infty(\mathscr{T}; \{\pm m_\beta\})$; we then call the regions $\{r \in \mathscr{T}: m(r) = \pm m_\beta\}$ the $\pm$ phases of m and define the interface of m as the boundary of the plus phase. On physical grounds the cost in free energy to create an interface is proportional to its area, the proportionality constant being the surface tension. Usually the surface tension is defined by imposing plus and minus conditions at the top and the bottom of a rectangular region (see, for instance, ref. 2), so that the interface is (on the average) flat and parallel to the bases of the rectangle. We will prove that the free energy is still proportional to the area times the surface tension even when the surface is not regular. Let $P(m)$ be the perimeter functional on $BV(\mathcal{T}; \{\pm m_\beta\})$ (the functions of bounded variation on $\mathcal{T}$ with values $\pm m_\beta$) that defines the area of the plus phase of m; see Appendix D. Then:

Theorem 2.5. There is $s_\beta > 0$, given in (2.22) below, so that the following holds. For all $u \in BV(\mathcal{T}; \{\pm m_\beta\})$ and all $R > 0$ there is $\zeta_\gamma \to 0^+$ as $\gamma \to 0^+$ so that

$$\lim_{\gamma \to 0^+} -\gamma^d \varepsilon^{d-1} \log[\mu_{\beta,\gamma,\varepsilon}(\mathcal{P}_{R,\zeta_\gamma,\gamma}(u))] = \beta s_\beta P(u) \tag{2.16}$$

Remarks. The number of spins in a spin configuration is proportional to $(\gamma\varepsilon)^{-d}$. Then, "in normal conditions," the large-deviation normalization factor is $(\gamma\varepsilon)^d$. In this sense the normalization in (2.16) is anomalous; the anomaly is due to the presence of a phase transition. Since there are two equilibrium magnetization values, $\pm m_\beta$, it is possible to have a nonconstant profile with equilibrium magnetization at all points: such profiles only cost a surface price. To get the prefactor $\gamma^d\varepsilon^{d-1}$ recall that the surfaces scale like ε^{1-d}, as $\mathcal{T}_\varepsilon = \varepsilon^{-1}\mathcal{T}$. As shown in the sequel, the thickness of the interface (in $\mathcal{T}_\varepsilon$) is of the order of unity, and thus the volume of the region around the interface scales also as ε^{1-d}. The number of spins in a region is proportional to the volume of the region times γ^{-d}; we thus obtain $\gamma^{-d}\varepsilon^{1-d}$, which is the inverse of the prefactor in (2.16).

To give an expression for s_β we need several intermediate definitions. We start by recalling a result proved in ref. 14:

Theorem 2.6. There is a unique function $\bar{m}: \mathbb{R} \to [-1, 1]$ such that $\bar{m}(0) = 0$,

$$\liminf_{s \to +\infty} \bar{m}(s) > 0, \qquad \limsup_{s \to -\infty} \bar{m}(s) < 0$$

and such that for all $s \in \mathbb{R}$

$$\bar{m}(s) = \tanh\{\beta \bar{J} * \bar{m}(s)\} \tag{2.17}$$

where $*$ denotes convolution and for every $s \in \mathbb{R}$

$$\bar{J}(s) := \int_{\mathbb{R}^{d-1}} dr\, J((s^2 + r^2)^{1/2}) \tag{2.18}$$

Moreover, $\bar{m} \in \mathscr{C}^{\infty}(\mathbb{R})$; it is antisymmetric, strictly increasing, and with asymptotic values at $\pm\infty$ equal to $\pm m_\beta$, to which it converges exponentially fast.

As we shall see, $\bar{m}$, called the instanton, describes the magnetization pattern at the interface. We next define the excess free energy functional $\mathscr{F}$:

Definition 2.7. Given a measurable set $\Lambda \subset \mathbb{R}^d$ and $m \in L^{\infty}(\Lambda; [-1, 1])$, we define the map $\mathscr{F}(m; \Lambda)$ with values in $[0, +\infty]$ as

$$\mathscr{F}(m; \Lambda) := \int_\Lambda dr\,[f(m(r)) - f(m_\beta)] + \frac{1}{4} \iint_{\Lambda \times \Lambda} dr\, dr'\, J(|r - r'|)[m(r) - m(r')]^2 \tag{2.19}$$

where, if $s \in [-1, 1]$, $f(s)$ and $i(s)$ are the free energy and the entropy density, namely

$$f(s) := -\frac{1}{2} s^2 - \beta^{-1} i(s) \tag{2.20}$$

$$i(s) := -\frac{1+s}{2} \log\left(\frac{1+s}{2}\right) - \frac{1-s}{2} \log\left(\frac{1-s}{2}\right) \tag{2.21}$$

If Λ is a torus, $|r - r'|$ is the distance between r and r' in the torus. We further set $\mathscr{F}(m) := \mathscr{F}(m; \mathbb{R}^d)$ and $\mathscr{F}_\varepsilon(m) := \mathscr{F}(m; \mathscr{T}_\varepsilon)$.

We call $\bar{\mathscr{F}}(m)$ the $d = 1$ version of $\mathscr{F}(m)$ with J replaced by $\bar{J}$; see (2.18). Together with Theorem 2.5 we shall prove that

$$s_\beta = \bar{\mathscr{F}}(\bar{m}) \tag{2.22}$$

The proof of Theorem 2.5 starts from a large-deviation estimate for the Gibbs measure. Because of the assumption on the size of the region, we are essentially reduced to the case considered by Eisele and Ellis,[16] and the large-deviation rate function is the functional $\mathscr{F}_\varepsilon$.

Given $m \in L^{\infty}(\mathscr{T}_\varepsilon; [-1, 1])$, we set

$$u(r) := m(\varepsilon^{-1} r), \qquad r \in \mathscr{T}$$

and define the functional

$$F_\varepsilon(u) := \varepsilon^{d-1} \mathscr{F}_\varepsilon(m) \tag{2.23}$$

on $L^\infty(\mathscr{T}; [-1, 1])$; thus

$$F_\varepsilon(u) = \varepsilon^{-1} \int_{\mathscr{T}} dr\, [f(u(r)) - f(m_\beta)] + \frac{\varepsilon}{4} \iint_{\mathscr{T} \times \mathscr{T}} dr\, dr'\, J_\varepsilon(|r - r'|) \left[\frac{u(r) - u(r')}{\varepsilon} \right]^2 \quad (2.24)$$

where

$$J_\varepsilon(|r|) := \varepsilon^{-d} J(\varepsilon^{-1} |r|) \quad (2.25)$$

In the next section we will prove that Theorem 2.5 follows by proving that $\{F_\varepsilon\}$ Γ-converges to $F(u) := s_\beta P(u)$ as $\varepsilon \to 0^+$. This means that given any $u \in BV(\mathscr{T}; \{\pm m_\beta\})$, the following hold:

1. For any family $\{u_\varepsilon\} \subseteq L^\infty(\mathscr{T}; [-1, 1])$ that converges in $L^1(\mathscr{T}; [-1, 1])$ to u as $\varepsilon \to 0^+$ we have

$$\liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) \geqslant F(u) \quad (2.26)$$

2. There exists a sequence $\{u_\varepsilon\} \subseteq L^\infty(\mathscr{T}; [-m_\beta, m_\beta])$ that converges in $L^1(\mathscr{T}; [-1, 1])$ to u as $\varepsilon \to 0^+$ and such that

$$\lim_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) = F(u) \quad (2.27)$$

Since J_ε in (2.25) is an approximated delta function, it might look reasonable to replace the second integral in (2.24) by

$$D \frac{\varepsilon}{4} \int_{\mathscr{T}} dr\, |\nabla u|^2, \qquad \text{where} \quad D := \int_{\mathbb{R}^d} dr\, J(|r|)\, r^2 \quad (2.28)$$

We then obtain the classical example of functionals that Γ-converge to $P(u)$ (modulo the constant factor $\sqrt{D/2}$), as conjectured by De Giorgi and Franzoni in 1975[7] and proved by Modica and Mortola.[26] However, the constant is not the surface tension of our model!

In Section 4 we will prove that $\{F_\varepsilon\}$ Γ-converges to $s_\beta P(u)$.

3. REDUCTION TO A VARIATIONAL PRINCIPLE

In the first part of this section we prove a relation between the Gibbs probability $\mu_{\beta,\gamma,\varepsilon}$ and the functional $\mathscr{F}_\varepsilon(m)$ by showing that, for small γ, $\mu_{\beta,\gamma,\varepsilon}$ is well approximated by

$$\mu_{\beta,\gamma,\varepsilon}(\cdot) \approx \exp[-\beta \gamma^{-d} \mathscr{F}_\varepsilon(\cdot)] \quad (3.1)$$

The sense in which (3.1) holds is specified in Lemma 3.2 below. We can already say, however, that (3.1) will not refer to single-spin configurations, for which it is not valid, but rather to a coarse-grained version of the configuration itself that we will define in the sequel.

In the second part of the section we will see, with the help of (3.1), that the problem of computing a probability may be reduced to finding minima of the functional $\mathscr{F}_\varepsilon$. In this way we will relate the proof of Theorem 2.5 to the Γ-convergence problem stated at the end of Section 2.

We begin with a few definitions aimed at introducing the basic notion of coarse-grained configurations.

3.1. The Partition $\mathscr{Q}_\gamma$

Recalling the definition of α in (2.9), we choose $\delta > 0$ so that

$$d\alpha < \delta < 1 - \alpha \tag{3.2}$$

and call $\mathscr{Q}_\gamma$ the partition $\mathscr{Q}^{(k)}$ of Definition 2.1a with $k = n_\gamma$ and

$$n_\gamma := \left\lceil \frac{\delta \log(\gamma^{-1})}{\log 2} \right\rceil, \qquad \text{so that} \quad \gamma^\delta \leqslant 2^{-n_\gamma} < 2\gamma^\delta \tag{3.3}$$

3.2. The Coarse-Grained Configurations

Recalling that $C^{(k)}(r)$ is the atom of $\mathscr{Q}^{(k)}$ that contains r, we let $\pi^{(k)}$ be the map from $L^\infty(\mathbb{R}^d)$ into itself defined by

$$\pi^{(k)} f(r) := \frac{1}{|C^{(k)}|} \int_{C^{(k)}(r)} dr'\, f(r') \tag{3.4}$$

We then set $\pi_\gamma := \pi^{(n_\gamma)}$, $s^{(k)} := \pi^{(k)} \sigma_\gamma$ and $s_\gamma := \pi_\gamma \sigma_\gamma$, where σ_γ is a spin configuration (Definition 2.1b); s_γ will be called a coarse-grained (spin) configuration.

3.3. Weights and Energies of the Coarse-Grained (Spin) Configurations

The weight $W(s_\gamma)$ of the coarse-grained configuration s_γ is the number of spin configurations σ_γ such that $\pi_\gamma \sigma_\gamma = s_\gamma$.

We denote by $H_\varepsilon(\cdot) := H(\cdot\,; \mathscr{T}_\varepsilon)$; see (2.3). The energy of a coarse-grained configuration s_γ is then denoted by $H_\varepsilon(s_\gamma) = H(s_\gamma; \mathscr{T}_\varepsilon)$.

We finally write, by an abuse of notation,

$$\mu_{\beta,\gamma,\varepsilon}(s_\gamma)=\mu_{\beta,\gamma,\varepsilon}(\{\sigma_\gamma:\pi_\gamma\sigma_\gamma=s_\gamma\}) \tag{3.5}$$

namely $\mu_{\beta,\gamma,\varepsilon}(s_\gamma)$ is the probability of all spin configurations σ_γ whose coarse-grained image is s_γ.

In the next lemma we relate the energy and the weight of the coarse-grained configurations, respectively, to the original energy of the spin configurations and to the entropy functional I. The latter is defined, for any given bounded measurable region Λ in $\mathbb{R}^d$, as the functional $I(m;\Lambda)$ on $L^\infty(\Lambda;[-1,1])$:

$$I(m;\Lambda):=\int_\Lambda dr\, i(m(r)) \tag{3.6}$$

with $i(m)$ as in (2.21). We set $I_\varepsilon(m):=I(m;\mathscr{T}_\varepsilon)$.

Lemma 3.1. There are positive constants c_1, c_2, c_3, and c_4 such that for all spin configurations σ_γ, all $r\in\mathbb{R}^d$, and all $R>0$

$$\left|⨍_{B_R(r)} dr'\,\sigma_\gamma(r')-⨍_{B_R(r)} dr'\,\pi_\gamma\sigma_\gamma(r')\right|\leqslant c_1R^{-1}\gamma^\delta \tag{3.7}$$

Moreover,

$$|H_\varepsilon(\sigma_\gamma)-H_\varepsilon(\pi_\gamma\sigma_\gamma)|\leqslant c_2\gamma^\delta\varepsilon^{-d} \tag{3.8}$$

and for all coarse-grained configurations s_γ

$$|\log W(s_\gamma)-\gamma^{-d}I_\varepsilon(s_\gamma)|\leqslant c_3(\gamma\varepsilon)^{-d}\gamma^{2(1-\delta)d}\log(\gamma^{-1}) \tag{3.9}$$

Finally, denoting by N_γ the total number of distinct coarse-grained configurations s_γ, we have

$$\log N_\gamma\leqslant c_4(\gamma\varepsilon)^{-d}\gamma^{(1-\delta)d}\log(\gamma^{-1}) \tag{3.10}$$

We prove the lemma in Appendix A.

The functionals I_ε and H_ε are related to $\mathscr{F}_\varepsilon$ in a simple way: by expanding the square in the last term of (2.19), recalling the definition (2.20) of $f(m)$, and using (2.2), we get

$$\mathscr{F}_\varepsilon(m)=[H_\varepsilon(m)-\beta^{-1}I_\varepsilon(m)]-[H_\varepsilon(\hat{m}_\beta)-\beta^{-1}I_\varepsilon(\hat{m}_\beta)]$$
$$\text{where}\quad \hat{m}_\beta(r)\equiv m_\beta,\quad r\in\mathscr{T}_\varepsilon \tag{3.11}$$

Then, as a corollary of Lemma 3.1, we have the following version of (3.1).

Lemma 3.2. There is a constant $C_1 > 0$ so that for any coarse-grained spin configuration s_γ

$$|\log \mu_{\beta,\gamma,\varepsilon}(s_\gamma) + \beta\gamma^{-d}\mathscr{F}_\varepsilon(s_\gamma)| \leqslant C_1(\gamma\varepsilon)^{-d}\{\gamma^{(1-\delta)d}\log(\gamma^{-1}) + \gamma^\delta\} \quad (3.12)$$

Proof. By (3.5) and (2.4)

$$\mu_{\beta,\gamma,\varepsilon}(s_\gamma) = \sum_{\pi_\gamma\sigma_\gamma = s_\gamma} \frac{1}{Z_{\beta,\gamma,\varepsilon}} \exp[-\beta\gamma^{-d}H_\varepsilon(\sigma_\gamma)]$$

where $Z_{\beta,\gamma,\varepsilon}$ denotes the partition function in the torus $\mathscr{T}_\varepsilon$. Using (3.8), we write

$$\begin{aligned}\sum_{\pi_\gamma\sigma_\gamma = s_\gamma} &\exp[-\beta\gamma^{-d}H_\varepsilon(\sigma_\gamma)] \\ &\geqslant \sum_{\pi_\gamma\sigma_\gamma = s_\gamma} \exp[-\beta\gamma^{-d}H_\varepsilon(s_\gamma) - \beta c_2\gamma^\delta(\gamma\varepsilon)^{-d}] \\ &= \exp[-\beta\gamma^{-d}H_\varepsilon(s_\gamma) - \beta c_2\gamma^\delta(\gamma\varepsilon)^{-d}]\ W(s_\gamma)\end{aligned}$$

(and similarly for the upper bound). We call

$$\psi_\gamma = (\gamma\varepsilon)^{-d}\{c_2\gamma^\delta + c_3\gamma^{(1-\delta)d}\log(\gamma^{-1})\} \quad (3.13)$$

and we get, using (3.9),

$$\begin{aligned}&\exp\{-\gamma^{-d}[\beta H_\varepsilon(s_\gamma) - I_\varepsilon(s_\gamma)] - \psi_\gamma\} \\ &\quad\leqslant \sum_{\pi_\gamma\sigma_\gamma = s_\gamma} \exp[-\beta\gamma^{-d}H_\varepsilon(\sigma_\gamma)] \leqslant \exp\{-\gamma^{-d}[\beta H_\varepsilon(s_\gamma) - I_\varepsilon(s_\gamma)] + \psi_\gamma\}\end{aligned} \quad (3.14)$$

Calling $m_{\beta,\gamma}$ the closest number to m_β which belongs to the range of $s_\gamma(r)$, we have

$$|m_\beta - m_{\beta,\gamma}| \leqslant c_5\gamma^{(1-\delta)d} \quad (3.15)$$

and setting $\hat{m}_{\beta,\gamma}(r) \equiv m_{\beta,\gamma}$, we have

$$|[\beta H_\varepsilon(\hat{m}_{\beta,\gamma}) - I_\varepsilon(\hat{m}_{\beta,\gamma})] - [\beta H_\varepsilon(\hat{m}_\beta) - I_\varepsilon(\hat{m}_\beta)]| \leqslant c_6(\gamma\varepsilon)^{-d}\gamma^{(1-\delta)d} \quad (3.16)$$

Therefore, using the lower bound in (3.14), we obtain

$$\begin{aligned}Z_{\beta,\gamma,\varepsilon} &\geqslant \sum_{\pi_\gamma\sigma_\gamma = \hat{m}_{\beta,\gamma}} \exp[-\beta\gamma^{-d}H_\varepsilon(\sigma_\gamma)] \\ &\geqslant \exp\{-\gamma^{-d}[\beta H_\varepsilon(\hat{m}_\beta) - I_\varepsilon(\hat{m}_\beta)] - \psi_\gamma - c_6(\gamma\varepsilon)^{-d}\gamma^{(1-\delta)d}\}\end{aligned} \quad (3.17)$$

Hence, recalling (3.11), we find

$$\log(\mu_{\beta,\gamma,\varepsilon}(s_\gamma)) \leqslant -\gamma^{-d}\beta\mathscr{F}_\varepsilon(s_\gamma) + 2\psi_\gamma + c_6(\gamma\varepsilon)^{-d}\gamma^{(1-\delta)d} \tag{3.18}$$

For the upper bound we write

$$\begin{aligned} Z_{\beta,\gamma,\varepsilon} &= \sum_{\{s_\gamma\}} \sum_{\pi_\gamma\sigma_\gamma = s_\gamma} \exp[-\beta\gamma^{-d}H_\varepsilon(\sigma_\gamma)] \\ &\leqslant \sum_{\{s_\gamma\}} \exp\{-\gamma^{-d}[\beta H_\varepsilon(s_\gamma) - I_\varepsilon(s_\gamma)] + \psi_\gamma\} \end{aligned} \tag{3.19}$$

Since for all s_γ

$$\beta H_\varepsilon(s_\gamma) - I_\varepsilon(s_\gamma) \geqslant \beta H_\varepsilon(\hat{m}_\beta) - I_\varepsilon(\hat{m}_\beta)$$

we get

$$\begin{aligned} \log Z_{\beta,\gamma,\varepsilon} &\leqslant -\gamma^{-d}[\beta H_\varepsilon(\hat{m}_\beta) - I_\varepsilon(\hat{m}_\beta)] + \psi_\gamma + \log N_\gamma \\ &\leqslant -\gamma^{-d}[\beta H_\varepsilon(\hat{m}_\beta) - I_\varepsilon(\hat{m}_\beta)] + \psi_\gamma + c_4(\gamma\varepsilon)^{-d}\gamma^{(1-\delta)d}\log(\gamma^{-1}) \end{aligned} \tag{3.20}$$

having used (3.10) in the last inequality. In conclusion,

$$\log(\mu_{\beta,\gamma,\varepsilon}(s_\gamma)) \geqslant -\gamma^{-d}\beta\mathscr{F}_\varepsilon(s_\gamma) - 2\psi_\gamma - c_4(\gamma\varepsilon)^{-d}\gamma^{(1-\delta)d}\log(\gamma^{-1}) \tag{3.21}$$

The bounds (3.18) and (3.21) together with the obvious inequalities

$$\gamma^{(1-\delta)d}\log(\gamma^{-1}) \geqslant \gamma^{(1-\delta)d}$$

prove (3.12). The proof of the lemma is complete. ∎

By using Lemma 3.2, it is easy to prove Theorem 2.3:

Proof of Theorem 2.3. Recalling the definition (3.4) of $\pi^{(k)}$, for any $\zeta^* > 0$ we set

$$\begin{aligned} \mathscr{A} = \{&m \in L^\infty(\mathscr{T}_\varepsilon; [-1, 1]): \\ &\text{there is } r \in \mathbb{R}^d \text{ such that } ||\pi^{(k)}m(r)| - m_\beta| \geqslant \zeta^*\} \end{aligned} \tag{3.22}$$

$$\begin{aligned} \mathscr{B} = \{&m \in L^\infty(\mathscr{T}_\varepsilon; [-1, 1]) \cap \mathscr{A}^c: \\ &\text{there are } r', r'' \in \mathbb{R}^d \text{ such that } |\pi^{(k)}m(r') - m_\beta| < \zeta^* \\ &\text{and } |\pi^{(k)}m(r'') + m_\beta| < \zeta^*\} \end{aligned} \tag{3.23}$$

We need the following lemma:

Lemma. For any k large enough and any ζ^* small enough there is a constant $c_7 > 0$ independent of ε such that

$$\inf_{m \in \mathscr{A} \cup \mathscr{B}} \mathscr{F}_\varepsilon(m) \geqslant c_7 \tag{3.24}$$

Proof of the Lemma. Suppose $m \in \mathscr{A}$. Call $C = C^{(k)}(r)$ the cube in $\mathscr{Q}^{(k)}$ that contains the point r appearing in (3.22). We set

$$\zeta := \frac{\zeta^*}{2}, \qquad \alpha := \frac{\zeta\,|C|}{8}$$

Calling

$$\Lambda_\pm := \{ r \in C \colon |m(r) \mp m_\beta| \leqslant \zeta \}; \qquad \Lambda_0 := C \backslash (\Lambda_+ \cup \Lambda_-)$$

we first consider the case when $|\Lambda_0| > \alpha$. Then there is a function $\kappa(\zeta) > 0$ such that

$$\mathscr{F}_\varepsilon(m) \geqslant \int_{\Lambda_0} dr\,[f(m(r)) - f(m_\beta)] \geqslant \kappa(\zeta)\,\alpha$$

When $|\Lambda_0| \leqslant \alpha$ we may also suppose, without loss of generality, that $|\Lambda_+| \geqslant |\Lambda_-|$. By definition

$$\int_C dr\, m(r) < (m_\beta - \zeta^*)\,|C|$$

hence, since $m \geqslant -1$ on Λ_0, we find

$$|\Lambda_+|(m_\beta - \zeta) - |\Lambda_0| + |\Lambda_-|(-m_\beta - \zeta) < (m_\beta - \zeta^*)\,|C|$$

Substituting $|C| = |\Lambda_+| + |\Lambda_-| + |\Lambda_0|$, we get

$$|\Lambda_-|(2m_\beta - \zeta^* + \zeta) > |\Lambda_+|(\zeta^* - \zeta) - 2\,|\Lambda_0|$$

Since $|\Lambda_+| \geqslant (|C| - \alpha)/2$, we have

$$2\,|\Lambda_-| > \frac{|C| - \alpha}{2}\zeta - 2\alpha = \frac{|C|\,\zeta}{4} - \frac{\alpha\zeta}{2} \geqslant \frac{|C|\,\zeta}{8}$$

having supposed that $m_\beta + \zeta \leqslant 1$. Then

$$|A_+| \geqslant |A_-| \geqslant \frac{|C|\,\zeta}{16}$$

We can then conclude that in C there are two sets $A_\pm$, $|A_\pm| = |C|\,\zeta/16$, where $m(r)$ is respectively close to $\pm m_\beta$ by at most ζ; moreover, by the isomorphism of Lebesgue measures,[29] there is a one-to-one map ψ from A_+ onto A_- which preserves the Lebesgue measure.

If $|C|$ is small enough (i.e., k large), there is another cube C' in $\mathscr{T}_\varepsilon$ and $a > 0$ such that $J(|r - r'|) \geqslant a$ for all $r \in C$ and $r' \in C'$. We can then bound

$$\mathscr{F}_\varepsilon(m) \geqslant \frac{a}{4} \int_{C'} dr' \int_{A_+ \cup A_-} dr\, [m(r') - m(r)]^2$$

We write the integral over r as

$$\int_{A_+} dr\, \{[m(r') - m(r)]^2 + [m(r') - m(\psi(r))]^2$$

$$\geqslant \tfrac{1}{2} \int_{A_+} dr\, [m(\psi(r)) - m(r)]^2$$

thus proving the bound (3.24) limited to $m \in \mathscr{A}$.

If $m \in \mathscr{B}$, by definition it is not in $\mathscr{A}$; then, without loss of generality, we may suppose that the closures of the two cubes of $\mathscr{Q}^{(k)}$ that contain r' and r'' [see (3.23)] have nonempty intersection. We can then apply the same argument used for $\mathscr{A}$ and the lemma is proved. ∎

We proceed with the proof of Theorem 2.3. Using (3.12), we have

$$\mu_{\beta,\gamma,\varepsilon}(\{s_\gamma^{(k)} \in \mathscr{A} \cup \mathscr{B}\}) \leqslant N_\gamma \exp\{-\beta c_7 \gamma^{-d} + (\gamma\varepsilon)^{-d} C_1[\gamma^\delta + \gamma^{(1-\delta)d} \log(\gamma^{-1})]\} \quad (3.25)$$

which vanishes as $\gamma \to 0^+$ because of (3.10) and (3.3).

We have thus proved that the union of the two sets

$$\{|s_\gamma^{(k)}(r) - m_\beta| < \zeta^*\}, \qquad \{|s_\gamma^{(k)}(r) + m_\beta| < \zeta^*\} \quad (3.26)$$

has probability going to 1 as $\gamma \to 0^+$, for any given k.

On the other hand, similarly to (3.7), we have, for any $r \in \mathbb{R}^d$ and $R > 0$,

$$\left| ⨍_{B_R(r)} dr'\, \sigma_\gamma(r') - ⨍_{B_R(r)} dr'\, \pi^{(k)} \sigma_\gamma(r') \right| \leqslant c_8 R^{-1} 2^{-k} \quad (3.27)$$

Therefore, if $s_\gamma^{(k)}$ is in the first set in (3.26), then σ_γ is in the set $\mathscr{P}^+_{R,\zeta,\gamma}$ of Definition 2.2 with $\zeta = \zeta^* + c_8 R^{-1} 2^{-k}$. Since a similar property holds for $\mathscr{P}^-_{R,\zeta,\gamma}$, we conclude that the statement in Theorem 2.3 is verified for the special case when ζ_γ is equal to a constant independent of γ. Because this holds for any value of the constant, by a diagonalization procedure we conclude that there is a sequence $\{\zeta_\gamma\}$ infinitesimal as $\gamma \to 0^+$ for which the assertion remains true. The proof of Theorem 2.3 is thus completed. ∎

In the remaining part of this section we will use Lemmas 3.1 and 3.2 to reduce the proof of Theorem 2.5 to the Γ-convergence problem for the functionals F_ε stated at the end of Section 2.

Let

$$m_\varepsilon^*(r) := u_\varepsilon^*(\varepsilon r), \qquad r \in \mathscr{T}_\varepsilon \tag{3.28}$$

where $\{u_\varepsilon^*\}$ is the minimizing sequence in condition 2 for Γ-convergence. Let

$$\zeta_\gamma^* := \int_{\mathscr{T}} dr\, |u_\varepsilon^*(r) - u(r)| \tag{3.29}$$

(recall that $\varepsilon^{-1} = [\gamma^{-\alpha}]$) and let $R > 0$; we specify the sequence ζ_γ in Theorem 2.5 as

$$\zeta_\gamma := \zeta_\gamma^* + c_1 R^{-1} \gamma^\delta \tag{3.30}$$

where c_1 is defined in Lemma 3.1; see (3.7). The reason for this choice will become clear in the sequel.

Observe that given ζ and R positive and setting

$$\zeta^\pm := \zeta \pm c_1 R^{-1} \gamma^\delta, \qquad \text{thus} \quad \zeta_\gamma^- = \zeta_\gamma^* \tag{3.31}$$

by (3.7) we have

$$\begin{aligned} \mu_{\beta,\gamma,\varepsilon}(\{\pi_\gamma \sigma_\gamma \in \mathscr{P}_{R,\zeta_\gamma^*,\gamma}(m)\}) &\leqslant \mu_{\beta,\gamma,\varepsilon}(\{\sigma_\gamma \in \mathscr{P}_{R,\zeta_\gamma,\gamma}(m)\}) \\ &\leqslant \mu_{\beta,\gamma,\varepsilon}(\{\pi_\gamma \sigma_\gamma \in \mathscr{P}_{R,\zeta_\gamma^+,\gamma}(m)\}) \end{aligned} \tag{3.32}$$

The Upper Bound. By (3.12)

$$\begin{aligned} &\log[\mu_{\beta,\gamma,\varepsilon}(\{\sigma_\gamma \in \mathscr{P}_{R,\zeta_\gamma,\gamma}(m)\})] \\ &\quad \leqslant -\beta\gamma^{-d} \inf_{s_\gamma \in \mathscr{P}_{R,\zeta_\gamma^+,\gamma}(m)} \mathscr{F}_\varepsilon(s_\gamma) \\ &\qquad + \log N_\gamma + (\gamma\varepsilon)^{-d}\, C_1\{\gamma^{(1-\delta)d} \log(\gamma^{-1}) + \gamma^\delta\} \end{aligned} \tag{3.33}$$

We multiply both sides by $\gamma^d \varepsilon^{d-1}$ and let $\gamma \to 0^+$. By (3.10)

$$\lim_{\gamma \to 0^+} \gamma^d \varepsilon^{d-1} \log N_\gamma = 0$$

because $\gamma^{(1-\delta)d} \varepsilon^{-1} = \gamma^{(1-\delta)d-\alpha}$ and by (3.2), $(1-\delta)\, d > d\alpha \geqslant \alpha$. Similarly, the last term in (3.33) also vanishes, after having been multiplied by $\gamma^d \varepsilon^{d-1}$. Thus, supposing the validity of condition 1 of Γ-convergence, by (2.26) we get

$$\limsup_{\gamma \to 0^+} \gamma^d \varepsilon^{d-1} \log[\mu_{\beta,\gamma,\varepsilon}(\{\sigma_\gamma \in \mathscr{P}_{R,\zeta_\gamma,\gamma}(m)\})] \leqslant -\beta F(m) \tag{3.34}$$

The Lower Bound. We use (3.12), as in the upper bound, to get

$$\begin{aligned}\log[\mu_{\beta,\gamma,\varepsilon}(\{\sigma_\gamma \in \mathscr{P}_{R,\zeta_\gamma,\gamma}(m)\})]& \\ \geqslant -\beta\gamma^{-d} \inf_{s_\gamma \in \mathscr{P}_{R,\zeta^*_\gamma,\gamma}(m)} &\mathscr{F}_\varepsilon(s_\gamma) \\ -(\gamma\varepsilon)^{-d}\, C_1\{\gamma^{(1-\delta)d} &\log(\gamma^{-1}) + \gamma^\delta\}\end{aligned} \tag{3.35}$$

We have already seen in the proof of the upper bound that the last term in (3.35) multiplied by $\gamma^d \varepsilon^{d-1}$ vanishes as $\gamma \to 0^+$. We would like to use condition 2 of the Γ-convergence, (2.27), for the first term on the right-hand side of (3.35), but the minimizing sequence m^*_ε is not necessarily a coarse-grained configuration. However, by (3.11), we have

$$\mathscr{F}_\varepsilon(m^*_\varepsilon) = [H_\varepsilon(m^*_\varepsilon) - \beta^{-1} I_\varepsilon(m^*_\varepsilon)] - [H_\varepsilon(\hat{m}_\beta) - \beta^{-1} I_\varepsilon(\hat{m}_\beta)] \tag{3.36}$$

Then, similarly to (3.8),

$$\gamma^{-d}\, |H_\varepsilon(m^*_\varepsilon) - H_\varepsilon(\pi_\gamma m^*_\varepsilon)| \leqslant c'_2 \gamma^\delta (\gamma\varepsilon)^{-d} \tag{3.37}$$

and, by the concavity of the entropy $i(\cdot)$ [see (2.21)],

$$-I_\varepsilon(m^*_\varepsilon) \geqslant -I_\varepsilon(\pi_\gamma m^*_\varepsilon) \tag{3.38}$$

$\pi_\gamma m^*_\varepsilon$ may not yet be a coarse-grained function, but, recalling (3.15), there is a coarse-grained function $s^*_\gamma(r)$ such that

$$\sup_{r \in \mathscr{T}_\varepsilon} |\pi_\gamma m^*_\varepsilon(r) - s^*_\gamma(r)| \leqslant c'_5 \gamma^{(1-\delta)d} \tag{3.39}$$

Since the minimizing sequence $\{m^*_\varepsilon\}$ has values in $[-m_\beta, m_\beta]$ (see the end of Section 4), we have

$$|I_\varepsilon(\pi_\gamma m^*_\varepsilon) - I_\varepsilon(s^*_\gamma)| \leqslant c_9 (\gamma\varepsilon)^{-d}\, \gamma^{(1-\delta)d} \tag{3.40}$$

By the continuity of the energy $H_\varepsilon(\cdot)$ we then get

$$\gamma^{-d}\mathscr{F}_\varepsilon(m_\varepsilon^*) \geqslant \gamma^{-d}\mathscr{F}_\varepsilon(s_\gamma^*) - c_2'\gamma^\delta(\gamma\varepsilon)^{-d} + c_{10}(\gamma\varepsilon)^{-d}\gamma^{(1-\delta)d} \tag{3.41}$$

Thus, going back to the first term on the right-hand side of (3.35), we have

$$-\beta\gamma^{-d} \inf_{s_\gamma \in \mathscr{P}_{R,\zeta_\gamma^*,\gamma}(m)} \mathscr{F}_\varepsilon(s_\gamma)$$
$$\geqslant -\beta\gamma^{-d}\mathscr{F}_\varepsilon(m_\varepsilon^*) - c_2'\gamma^\delta(\gamma\varepsilon)^{-d} - c_{10}(\gamma\varepsilon)^{-d}\gamma^{(1-\delta)d} \tag{3.42}$$

We multiply both sides by $\gamma^d\varepsilon^{d-1}$ and let $\gamma \to 0^+$. Then by (3.42) and (2.27),

$$\liminf_{\gamma \to 0^+} \log[\mu_{\beta,\gamma,\varepsilon}(\{\sigma_\gamma \in \mathscr{P}_{R,\zeta_\gamma,\gamma}(m)\})] \geqslant -\beta F(m) \tag{3.43}$$

(3.43) and (3.34) prove (2.16), hence Theorem 2.5 is reduced to the proof of (2.26) and (2.27), namely to the Γ-convergence of $\{F_\varepsilon\}$ to F.

4. Γ-CONVERGENCE

In this section we prove that the functionals F_ε converge to F in the sense of Γ-convergence; see (2.26) and (2.27). While the proof of part 2 of Γ-convergence is standard, the proof of part 1 is less typical, due to the nonlocal structure of our functionals F_ε. As discussed at the end of Section 2, a local version of F_ε is the functional

$$M_\varepsilon(u) := \varepsilon \int_{\mathbb{R}^d} dr\ |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^d} dr\ w(u)$$

where, for simplicity, we set $w(s) := (1-s^2)^2$. The proof of the Γ-convergence of $\{M_\varepsilon\}$, considered by Modica and Mortola[26] and Modica,[25] exploits the following elementary inequality

$$M_\varepsilon(u) \geqslant 2\int_{\mathbb{R}^d} dr\ |\nabla u|[w(u)]^{1/2} =: L(u) \tag{4.1}$$

where ε has disappeared from the right-hand side. Then the family $\{L_\varepsilon\}$, $L_\varepsilon \equiv L$, Γ-converges to the lower semicontinuous envelope of the functional L. On the other hand, the minimizing sequence $\{u_\varepsilon\}$ realizes, roughly speaking, the equipartition of the energy, i.e.,

$$\varepsilon\ |\nabla u_\varepsilon|^2 = \frac{1}{\varepsilon}\ w(u_\varepsilon)$$

Hence $M_\varepsilon(u_\varepsilon) = L(u_\varepsilon)$.

In our case the inequality similar to (4.1) yields, for $d=1$, $J=\mathbf{1}_{\{r\in\mathbb{R}:\,|r|\leqslant 1\}}/2$ and with $[f(s)-f(m_\beta)]$ replaced by $w(s)$,

$$F_\varepsilon(u)\geqslant\frac{1}{\varepsilon}\int_{\mathbb{R}}dr\left(\frac{1}{8\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon}dr'\,|u(r)-u(r')|^2\right)^{1/2}[w(u)]^{1/2}=:L_\varepsilon(u)$$

However the Γ-limit of $\{L_\varepsilon\}$ (which corresponds in the previous case to the lower semicontinuous envelope of L) vanishes. Indeed, let us consider $u:=1$ on $[0,1]$ and $u:=-1$ elsewhere and choose $u_\varepsilon:=u$ for all ε. Then $L_\varepsilon(u_\varepsilon)=0$ for all ε, hence its limit vanishes.

Thus inequality (4.1) gives a trivial bound in our case and a different approach is required. The Γ-convergence of some nonlocal functionals has also been considered recently by Jost.[21]

After these introductory remarks we begin the proofs. We need some notation. Recall that given a set $E\subset\mathbb{R}^d$, $\mathbf{1}_E$ denotes the characteristic function of E, i.e., $\mathbf{1}_E(x)=1$ if $x\in E$ and $\mathbf{1}_E(x)=0$ otherwise. We call $R=B\times[-h,h]\subset\mathcal{T}$ a parallelepiped in $\mathcal{T}$ of height $2h$ and middle section B [which is supposed in turn to be a parallelepiped in $\mathbb{R}^{d-1}$ with $2(d-1)$ faces]. Thus B divides R into two parts that we call $R^\pm$ and, according to this choice, we set $\chi_R:=m_\beta(\mathbf{1}_{R^+}-\mathbf{1}_{R^-})$.

4.1. Proof of Condition 1 of Γ-Convergence

Theorem 4.1. Let $u\in BV(\mathcal{T};\{\pm m_\beta\})$ and let $\{u_\varepsilon\}\subset L^\infty(\mathcal{T};[-1,1])$ be a sequence converging to u in $L^1(\mathcal{T})$ as $\varepsilon\to 0^+$. Then

$$\liminf_{\varepsilon\to 0^+}F_\varepsilon(u_\varepsilon)\geqslant s_\beta P(u)$$

Proof. Let u and u_ε be as in the statement of the proposition. In Appendix D we prove that for any $\zeta'>0$ there are n disjoint parallelepipeds $R_1,\dots,R_n$ with bases [$(d-1)$-dimensional parallelepipeds] $B_1,\dots,B_n$, respectively, and equal height $2h$, so that

$$\frac{1}{h}\sum_{i=1}^{n}\int_{R_i}dr\,|\chi_{R_i}-u|<\zeta',\qquad\left|\sum_{i=1}^{n}|B_i|-P(u)\right|<\zeta'\tag{4.2}$$

By Proposition 4.2 below there exists an absolute constant $c>0$ such that for any $\zeta>0$

$$\liminf_{\varepsilon\to 0^+}F_\varepsilon(u_\varepsilon;R_i)\geqslant(s_\beta-c\zeta)\,|B_i|-\frac{c}{h\zeta}\int_{R_i}dr\,|\chi_{R_i}-u|,\qquad i=1,\dots,n\tag{4.3}$$

By (2.24) we have $F_\varepsilon(u_\varepsilon) \geqslant \sum_{i=1}^n F_\varepsilon(u_\varepsilon; R_i)$. We then have, in view of (4.2),

$$\begin{aligned}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) &\geqslant \sum_{i=1}^n \liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon; R_i)\\ &\geqslant (s_\beta - c\zeta)\sum_{i=1}^n |B_i| - \frac{c}{h\zeta}\sum_{i=1}^n \int_{R_i} dr\, |\chi_{R_i} - u|\\ &\geqslant s_\beta P(u) - c\frac{\zeta'}{\zeta} + O(\zeta) + O(\zeta')\end{aligned}$$

Taking $\zeta' := \zeta^2$ and observing that the previous inequality holds for any $\zeta > 0$, the proof of the proposition is concluded. ∎

Proposition 4.2. There is a constant $c > 0$ so that the following holds. Let $R \subset \mathscr{T}$ be a parallelepiped with basis B and height $2h$. If $\{u_\varepsilon\} \subset L^\infty(\mathscr{T}; [-1, 1])$ converges in $L^1(\mathscr{T})$ to $u \in BV(\mathscr{T}; \{\pm m_\beta\})$, then for any $\zeta > 0$

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon; R) \geqslant (s_\beta - c\zeta)\,|B| - \frac{c}{h\zeta}\int_R dr\, |\chi_R - u| \tag{4.4}$$

Before proving this proposition we introduce some definitions which shall be useful in the sequel.

Recalling Definition 2.1a, we consider the partition $\mathscr{Q}^{(-1)}$ of $\mathbb{R}^d$ into cubes D of side 2 and denote by $D(r)$ the cube $D \in \mathscr{Q}^{(-1)}$ that contains the point r. Given $D \in \mathscr{Q}^{(-1)}$, we define the height $\lambda(D)$ of D as $\lambda(D) = 2n$, $n \in \mathbb{Z}$, if $2n$ is the smallest value of the coordinate r_d of the points $r \in D$. Let $R = B \times [-h, h]$, $R_\varepsilon := \varepsilon^{-1}R \subset \mathscr{T}_\varepsilon := \varepsilon^{-1}\mathscr{T}$, $B_\varepsilon := \varepsilon^{-1}B$, $R_\varepsilon^\pm := \varepsilon^{-1}R^\pm$. For simplicity we suppose that $R_\varepsilon^\pm$ is $\mathscr{Q}^{(-1)}$-measurable; see Definition 2.1a.

For any $m \in L^\infty(R_\varepsilon; [-m_\beta, m_\beta])$, any positive integer k, and $\zeta > 0$, we define the function $\eta = \eta_{m,k,\zeta}$ as

$$\eta(r) := \begin{cases} 1 & \text{if } \pi^{(k)}m \geqslant m_\beta - \zeta \text{ on } D(r)\\ -1 & \text{if } \pi^{(k)}m \leqslant -m_\beta + \zeta \text{ on } D(r)\\ 0 & \text{otherwise}\end{cases} \tag{4.5}$$

where $\pi^{(k)}$ is defined in (3.4).

We introduce now the important notion of cluster; see also Fig. 1.

Definition 4.3. Given $m \in L^\infty(\mathscr{T}_\varepsilon; [-1, 1])$, a cluster $\mathscr{D} = \mathscr{D}(m)$ is a maximal $*$-connected union of cubes in R_ε^+ where $\eta(\cdot) < 1$ (two cubes are $*$-connected if their closures have nonempty intersection).

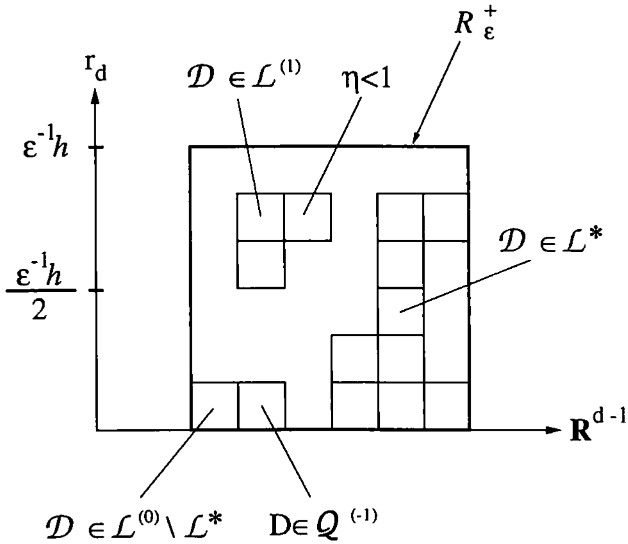

Fig. 1. The cubes $D \in \mathscr{Q}^{(-1)} \cap R_\varepsilon^+$ where $\eta < 1$. There are three clusters: the one on the upper left is in $\mathscr{L}^{(1)}$, the other two in $\mathscr{L}^{(0)}$; the one on the bottom left is in $\mathscr{L}^{(0)} \setminus \mathscr{L}^*$, the other one in $\mathscr{L}^*$, as its height exceeds $\varepsilon^{-1}h/2$.

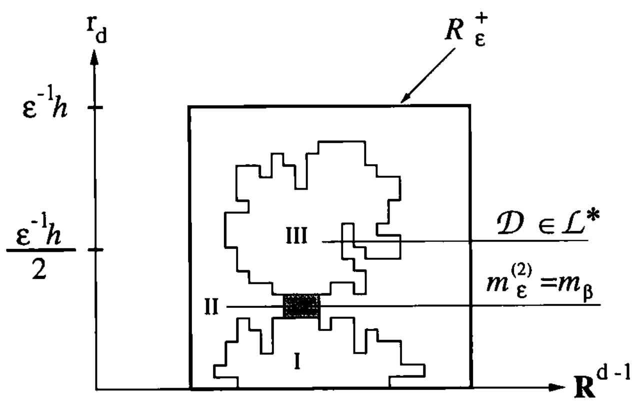

Fig. 2. A cluster $\mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)})$. $\mathscr{D}$ is divided into three parts, I, II, and III. The shadowed region, II, is the minimal section of $\mathscr{D}$. The modification $m_\varepsilon^{(2)}$ of $m_\varepsilon^{(1)}$ is equal everywhere to $m_\varepsilon^{(1)}$ except in II, where it has value m_β; thus $\eta_{m_\varepsilon^{(2)},k,\zeta} = 1$ on II, hence $\mathrm{III} \in \mathscr{L}^{(1)}(m_\varepsilon^{(2)})$ and $I \in \mathscr{L}^{(0)}(m_\varepsilon^{(2)})$, but $I \notin \mathscr{L}^*(m_\varepsilon^{(2)})$, which is thus empty.

$\mathscr{L}^0 = \mathscr{L}^0(m)$ is the set of clusters in R_ε^+ which have nonempty intersection with the basis B_ε of R_ε. The set $\mathscr{L}^1 = \mathscr{L}^1(m)$ collects the others.

Given a cluster $\mathscr{D}$, we define its height $\lambda(\mathscr{D})$ as

$$\lambda(\mathscr{D}) := \max\{\lambda(D) \colon D \in \mathscr{D}\}$$

and, for any even n, its section $B(n; \mathscr{D})$ at height n as

$$B(n; \mathscr{D}) := \{D \subset \mathscr{D} \colon \lambda(D) = n\} \tag{4.6}$$

Calling

$$n^* := \frac{\varepsilon^{-1}h}{2} \tag{4.7}$$

(having supposed for simplicity that $\varepsilon^{-1}h/2$ is even), we define

$$\mathscr{L}^* = \mathscr{L}^*(m) := \{\mathscr{D} \in \mathscr{L}^0 \colon \lambda(\mathscr{D}) \geqslant n^*\}$$

The minimal section $S(\mathscr{D})$ of $\mathscr{D} \in \mathscr{L}^*$ (see Fig. 2) is defined after setting

$$b(\mathscr{D}) := \min_{\substack{n \text{ even} \\ 0 < n \leqslant n^*}} |B(n, \mathscr{D})|$$

$$n^0 := \min\{n \text{ even},\ 0 < n \leqslant n^*,\ |B(n, \mathscr{D})| = b(\mathscr{D})\}$$

as

$$S(\mathscr{D}) := B(n^0, \mathscr{D}) \tag{4.8}$$

Finally, the symbols c, c' will denote absolute positive constants that may change from line to line.

Proof of Proposition 4.2. Set

$$m_\varepsilon(r) := u_\varepsilon(\varepsilon r), \qquad m(r) := u(\varepsilon r), \qquad r \in \mathscr{T}_\varepsilon \tag{4.9}$$

We will prove the proposition after many successive modifications of the function m_ε, each one either decreasing $\mathscr{F}_\varepsilon$ or increasing it by a "controlled" quantity. The first step is very simple; we just take

$$m_\varepsilon^{(1)}(r) := \begin{cases} \pm m_\beta & \text{if} \quad m_\varepsilon(r) \geqslant m_\beta, \quad \text{resp.,} \quad m_\varepsilon(r) \leqslant -m_\beta \\ m_\varepsilon(r) & \text{otherwise} \end{cases}$$

Then, obviously, $\mathscr{F}_\varepsilon(m_\varepsilon) \geqslant \mathscr{F}_\varepsilon(m_\varepsilon^{(1)})$.

Modulo rotation and reflection of the axes, we may suppose that the basis of R_ε is contained in the coordinate hyperplane $\{r_d = 0\}$, r_d denoting the last coordinate of r in $\mathbb{R}^d$, and that $R_\varepsilon^+ \subset \{r_d > 0\}$.

We next modify $m_\varepsilon^{(1)}$ in R_ε^+ (the modification in R_ε^- is similar and done later) by cutting the clusters in $\mathscr{L}^*(m_\varepsilon^{(1)})$ at their minimal section (see Fig. 2). The clusters are now defined by means of the function $\eta_{m_\varepsilon^{(1)},k,\zeta}$; see (4.5). Let

$$m_\varepsilon^{(2)} := \begin{cases} m_\beta & \text{on} \quad S(\mathscr{D}),\ \mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)}) \\ m_\varepsilon^{(1)} & \text{elsewhere on } R_\varepsilon \end{cases}$$

Then

$$\mathscr{F}(m_\varepsilon^{(1)}; R_\varepsilon) \geqslant \mathscr{F}(m_\varepsilon^{(2)}; R_\varepsilon) - c \sum_{\mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)})} |S(\mathscr{D})| \tag{4.10}$$

We shall see at the end of the proof of this proposition that the cost of this substitution, that is, the last term in (4.10), can be controlled in terms of the $L^1(R)$ norm of $\chi_R - u_\varepsilon$.

By construction, $\mathscr{L}^*(m_\varepsilon^{(2)}) = \varnothing$, so that the "dangerous clusters" that intersect the basis of R_ε, i.e., those in $\mathscr{L}^0(m_\varepsilon^{(2)})$, are all "low," in the sense that they do not reach the height n^*, which is half of the total height of R_ε^+. We next apply Theorem B.2 of Appendix B to modify $m_\varepsilon^{(2)}$ into a function $m_\varepsilon^{(3)}$, which, as we shall see, is positive on the clusters of $\mathscr{L}^1(m_\varepsilon^{(2)})$, where instead $m_\varepsilon^{(2)}$ may be negative: the proof exploits the fact that by definition all the clusters of $\mathscr{L}^1(m_\varepsilon^{(2)})$ are surrounded by cubes where $\eta(\cdot) = 1$. The precise statement can be found in Theorem B.2, which we apply in the present context with Δ, Γ, and Λ defined as follows:

$$\Delta := \bigcup \{\mathscr{D} : \mathscr{D} \in \mathscr{L}^1(m_\varepsilon^{(2)})\}$$

$$\Gamma := \{D \in \mathscr{Q}^{(-1)} \cap R_\varepsilon^+ : D \cap \Delta = \varnothing, \bar{D} \cap \Delta \neq \varnothing\}$$

$$\Lambda := R_\varepsilon \backslash (\Delta \cup \Gamma)$$

The parameters ζ and k of Theorem B.2 are the same parameters as in the definition of the functions η in (4.5); l is a number in $(0, 1)$ and the sequence $\{c_k\}$ [defined in (B.7)] is actually independent of ε (see the Remark at the end of the proof of Lemma B.1) and tends to 0 as $k \to +\infty$; finally we recall that $0 < \theta := \zeta + c_k < m_\beta$. We choose ζ and k so that $\theta = \zeta + c_k \leqslant m_\beta/4$.

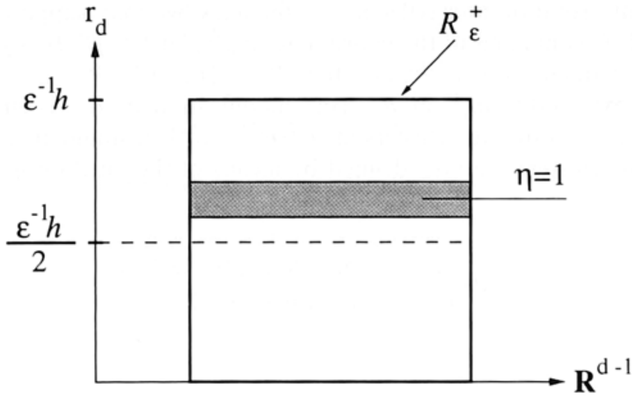

Fig. 3. $\mathscr{L}^{(1)}(m_\varepsilon^{(3)}) = \mathscr{L}^*(m_\varepsilon^{(3)}) = \varnothing$; therefore $\eta = 1$ on $\{\lambda(D) \geqslant n^* + 4\}$. In particular, $\eta = 1$ on the level $\lambda(D) = n^* + 4$, which is shaded. While the averages are $\pi_k m_\varepsilon^{(3)} \geqslant m_\beta - \theta$ above this level, this may not be so point by point. However, for the modification $m_\varepsilon^{(4)}$ of $m_\varepsilon^{(3)}$, $m_\varepsilon^{(4)} \geqslant m_\beta - \theta$ on $\{\lambda(D) > n^* + 4\}$.

Since $m_\varepsilon^{(2)}$ satisfies condition (B.4), i.e., $\pi^{(k)} m_\varepsilon^{(2)} \geqslant m_\beta - \zeta$ for all $r \in \Gamma$, then, by Theorem B.2, there exists $m_\varepsilon^{(3)} := (m_\varepsilon^{(2)})_{l,\theta}$ on R_ε with the following properties:

1. $\mathscr{F}(m_\varepsilon^{(2)}; R_\varepsilon) \geqslant \mathscr{F}(m_\varepsilon^{(3)}; R_\varepsilon)$.
2. $m_\varepsilon^{(3)} \geqslant m_\varepsilon^{(2)}$ on R_ε, and $m_\varepsilon^{(3)}(r) = m_\varepsilon^{(2)}(r)$ for all $r \in R_\varepsilon$ at distance not smaller than 1 from Δ.
3. $m_\varepsilon^{(3)} \geqslant m_\beta - \theta$ on Δ.

By construction, $-m_\beta \leqslant m_\varepsilon^{(3)} \leqslant m_\beta$, $\mathscr{L}^*(m_\varepsilon^{(3)}) = \varnothing$, and $m_\varepsilon^{(3)} \geqslant m_\beta - \theta$ on all clusters $\mathscr{D}$ with $\mathscr{D} \cap B_\varepsilon = \varnothing$ where $\eta_{m_\varepsilon^{(2)},k,\zeta} < 1$ (i.e., where $m_\varepsilon^{(2)}$ is "far" from the value m_β).

We redefine the clusters [see (4.5)] for $m_\varepsilon^{(3)}$ with ζ replaced by θ, i.e., by means of the function $\eta_{m_\varepsilon^{(3)},k,\theta}$. By an abuse of notation the new clusters are denoted by the same symbols as the old ones and we also denote θ again by ζ. By construction, if $\mathscr{D} \in \mathscr{L}^0(m_\varepsilon^{(3)})$, then $B(n, \mathscr{D}) = \varnothing$ for $n > n^*$; see (4.6) for notation.

We now modify $m_\varepsilon^{(3)}$ into $m_\varepsilon^{(4)}$ in such a way that $m_\varepsilon^{(4)} \geqslant m_\beta - \theta$ on all $D \in \mathscr{Q}^{(-1)}$ with $\lambda(D) \geqslant n^* + 4$; see Fig. 3. Precisely, we apply again Theorem B.2 with Δ, Γ, and Λ defined as follows:

$$\Delta := \{D \in \mathscr{Q}^{(-1)} \cap R_\varepsilon^+ : \lambda(D) \geqslant n^* + 4\}$$

$$\Gamma := \{D \in \mathscr{Q}^{(-1)} \cap R_\varepsilon^+ : \lambda(D) = n^* + 2\}$$

$$\Lambda := R_\varepsilon \backslash (\Delta \cup \Gamma)$$

[note that $\eta_{m_\varepsilon^{(2)},k,\zeta}(\cdot)=1$ on Γ, so that $m_\varepsilon^{(2)}$ satisfies condition (B.4)]. Then by Theorem B.2 there exists $m_\varepsilon^{(4)}$ on R_ε with the following properties:

1. $\mathscr{F}(m_\varepsilon^{(3)}; R_\varepsilon) \geqslant \mathscr{F}(m_\varepsilon^{(4)}; R_\varepsilon)$.
2. $m_\varepsilon^{(4)} \geqslant m_\varepsilon^{(3)}$ on R_ε, and $m_\varepsilon^{(4)} = m_\varepsilon^{(3)}$ on all D with $\lambda(D) < n^* + 2$.
3. $m_\varepsilon^{(4)} \geqslant m_\beta - \theta$ on Δ, i.e., on all D with $\lambda(D) \geqslant n^* + 4$.

We conclude this first part of the proof of Proposition 4.2 by introducing the function $m_\varepsilon^{(5)}$, obtained by repeating (with opposite sign) the modifications that led to $m_\varepsilon^{(4)}$ also in the lower part R_ε^- of the parallelepiped R_ε.

Summary of What Has Been Done so Far. There is a function $m_\varepsilon^{(5)}$ which is larger than $m_\beta - \theta$ for $r_d \geqslant n^* + 4$ and smaller than $-m_\beta + \theta$ for $r_d \leqslant -n^* - 4$, such that

$$\mathscr{F}(m_\varepsilon; R_\varepsilon) \geqslant \mathscr{F}(m_\varepsilon^{(5)}; R_\varepsilon) - c \sum_{\mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)})} |S(\mathscr{D})| \tag{4.11}$$

(here $\mathscr{L}^*$ is referred to the whole R_ε).

Let $K_\varepsilon := B_\varepsilon \times (-\infty, +\infty)$,

$$m_\varepsilon^{(6)} := \begin{cases} m_\varepsilon^{(5)} & \text{on } R_\varepsilon \\ \pm m_\beta & \text{on } K_\varepsilon^\pm \backslash R_\varepsilon^\pm \end{cases}$$

Then one can verify that

$$\mathscr{F}(m_\varepsilon^{(5)}; R_\varepsilon) \geqslant \mathscr{F}(m_\varepsilon^{(6)}; K_\varepsilon) - c\zeta\, |B_\varepsilon| \tag{4.12}$$

(choosing k so large that $c_k < \zeta$).

We would like to have $\mathscr{F}(m_\varepsilon^{(6)}; K_\varepsilon) \geqslant \mathscr{F}(m^*; K_\varepsilon)$ when m^* is the instanton $\bar{m}$ of Theorem 2.6 on each line parallel to the r_d axis of K_ε. However, we are not able to make an estimate of $\mathscr{F}(m_\varepsilon^{(6)}; K_\varepsilon)$ to verify the inequality. A possible way out would be to prove that the infimum of $\mathscr{F}(\cdot; K_\varepsilon)$ over all $m \in L^\infty(K_\varepsilon; [-m_\beta, m_\beta])$ having asymptotic values $\pm m_\beta$ for $r_d \to \pm\infty$ is just $\mathscr{F}(m^*; K_\varepsilon)$. This is, however, ruled out by the fact that m^* is not a critical point of $\mathscr{F}(\cdot; K_\varepsilon)$. In fact, all critical points of $\mathscr{F}(\cdot; K_\varepsilon)$ (see the Remark below) must satisfy

$$m(r) = \tanh\{\beta \tilde{J} * m(r)\}, \qquad \tilde{J} * m(r) := \int_{K_\varepsilon} dr'\, \tilde{J}(r, r')\, m(r') \tag{4.13}$$

where

$$\tilde{J}(r, r') := J(|r-r'|)\, \mathbf{1}_{\{r, r' \in K_\varepsilon\}} + j(r)\, \delta(r-r')$$
$$j(r) := \int_{\{r'' \notin K_\varepsilon\}} dr''\, J(|r-r''|) \tag{4.14}$$

and $\delta(r-r')$ is the delta function. Since $m^*(r) = \tanh\{\beta J * m^*(r)\}$ when the convolution is over the whole $\mathbb{R}^d$, in general $m^*(r) \neq \tanh\{\beta \tilde{J} * m^*(r)\}$ when the distance of r from ∂K_ε is smaller than 1.

Remark. The Euler–Lagrange equation for the functional $\mathscr{F}(m) := \mathscr{F}(m; \mathbb{R}^d)$ is

$$f'(m) + m = J * m \tag{4.15}$$

Recalling the definition of f in (2.20), we find that (4.15) becomes

$$-i'(m) = \beta J * m$$

Since $-i'(s) = \operatorname{arctanh}(s)$, one deduces that a function $m \in L^\infty(\mathbb{R}^d; (-1, 1))$ solves (4.15) if and only if

$$m = \tanh(\beta J * m)$$

To overcome the problems due to the presence of the delta function in the interaction, we introduce an auxiliary functional $\mathscr{F}^{(1)}(\cdot\,; K_\varepsilon) \leqslant \mathscr{F}(\cdot\,; K_\varepsilon)$ for which it is possible to prove that there is a unique critical point (a minimum) which is an instanton function close to m^*. This minimum is found by means of an auxiliary dynamics on $L^\infty(K_\varepsilon; [-1, 1])$ under which the functional $\mathscr{F}^{(1)}(\cdot\,; K_\varepsilon)$ is monotonic (nonincreasing). The minimum is then obtained as the limit point when $t \to +\infty$ of the orbit m_t that starts from $m_\varepsilon^{(6)}$. The analysis adapts to the present context results known in the literature.[13,14] Even though the main ideas are the same, the extension is not totally trivial and we will report some details in Appendix C. The dynamics here is merely a technical tool, but the evolution has an intrinsic interest with many significant applications in the physics of phase separation and mathematical interest in its own right; see, for instance, refs. 9–12.

As explained above, a way to avoid the delta function in (4.14) is to modify $\mathscr{F}(\cdot\,; K_\varepsilon)$ into another functional, $\mathscr{F}^{(1)}(\cdot\,; K_\varepsilon)$, which no longer produces the dangerous local term in the interaction. For any $m \in L^\infty(K_\varepsilon; [-1, 1])$ we set

$$\mathscr{F}^{(1)}(m; K_\varepsilon) := \int_{K_\varepsilon} dr\, [1 - j(r)][f(m(r)) - f(m_\beta)]$$
$$+ \tfrac{1}{4} \iint_{K_\varepsilon \times K_\varepsilon} dr\, dr'\, J(|r-r'|)[m(r) - m(r')]^2 \tag{4.16}$$

Obviously

$$\mathscr{F}^{(1)}(m; K_\varepsilon) \leqslant \mathscr{F}(m; K_\varepsilon) \tag{4.17}$$

and since

$$\int_{K_\varepsilon} dr\, [1 - j(r)]\, m(r)^2 = \iint_{K_\varepsilon \times K_\varepsilon} dr\, dr'\, m(r)^2\, J(|r - r'|)$$

we have, recalling (2.20),

$$\begin{aligned} \mathscr{F}^{(1)}(m; K_\varepsilon) = &-\beta^{-1} \int_{K_\varepsilon} dr\, [1 - j(r)]\, i(m(r)) \\ &- \tfrac{1}{2} \iint_{K_\varepsilon \times K_\varepsilon} dr\, dr'\, J(|r - r'|)\, m(r)\, m(r') - C^{(1)}(\beta) \end{aligned} \tag{4.18}$$

where $C^{(1)}(\beta)$ is the sum of the first two terms when $m(\cdot) \equiv m_\beta$.

By direct inspection $\mathscr{F}^{(1)}(\cdot; K_\varepsilon)$ does not increase along the solutions of the equation

$$\begin{aligned} \frac{dm_t(r)}{dt} &= -m_t(r) + \tanh\{\beta J^{(1)} * m_t(r)\} \\ J^{(1)} * m_t(r) &:= \int_{K_\varepsilon} dr'\, J^{(1)}(r, r')\, m_t(r') \end{aligned} \tag{4.19}$$

where

$$J^{(1)}(r, r') := \frac{1}{1 - j(r)}\, J(|r - r'|), \qquad \int_{K_\varepsilon} dr'\, J^{(1)}(r, r') = 1 \tag{4.20}$$

We show in Appendix C that there is an (instanton-like) function $m_\varepsilon^{(7)}$ on K_ε, which is a stationary solution of (4.19) in the whole K_ε. This solution is an antisymmetric function of r_d and there are $c > 0$ and c' independent of ε and of the section B_ε of the cylinder K_ε so that

$$m_\varepsilon^{(7)}(r) \geqslant m_\beta - c' e^{-c r_d}, \qquad r_d \geqslant 0 \tag{4.21}$$

Moreover, if m_t is the orbit starting from $m_\varepsilon^{(6)}$, then there is $r^0 = (0, r_d^0)$ so that

$$\lim_{t \to +\infty} m_t(r) = m_\varepsilon^{(7)}(r - r^0) \tag{4.22}$$

uniformly on the compact subsets of K_ε. By the lower semicontinuity of $\mathscr{F}^{(1)}(\cdot; K_\varepsilon)$ and its invariance under vertical translations,

$$\mathscr{F}^{(1)}(m_\varepsilon^{(6)}; K_\varepsilon) \geqslant \liminf_{t \to +\infty} \mathscr{F}^{(1)}((m_\varepsilon^{(6)})_t; K_\varepsilon) \geqslant \mathscr{F}^{(1)}(m_\varepsilon^{(7)}; K_\varepsilon) \tag{4.23}$$

By (4.11), (4.12), (4.17), and (4.23) we have

$$\begin{aligned}\mathscr{F}(m_\varepsilon; R_\varepsilon) &\geqslant \mathscr{F}^{(1)}(m_\varepsilon^{(6)}; K_\varepsilon) - c\zeta\,|B_\varepsilon| - c \sum_{\mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)})} |S(\mathscr{D})| \\ &\geqslant \mathscr{F}^{(1)}(m_\varepsilon^{(7)}; K_\varepsilon) - c\zeta\,|B_\varepsilon| - c \sum_{\mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)})} |S(\mathscr{D})| \end{aligned} \tag{4.24}$$

where $\mathscr{L}^*$ is referred to the whole R_ε.

Our last effort will be to replace $m_\varepsilon^{(7)}$ by m^*, which we recall is the instanton $\bar{m}$ of Theorem 2.6 on each line parallel to the r_d axis of K_ε, with the 0's belonging to the original middle section B_ε of R_ε. The natural way would be to prove that $m_\varepsilon^{(7)}$ is really close to m^* except possibly at points close to the boundaries and with height r_d not too large. Recall in fact that both $m_\varepsilon^{(7)}$ and m^* have the same asymptotic values, which they approach exponentially fast. However, although we believe this statement to be correct, we lack a proof. We will proceed by changing the functional and the corresponding dynamics with the introduction of "Neumann conditions" at the boundaries. After that, the above comparison will become easier. To construct the kernel $J^{(2)}$ we need to smooth the "corners" of B_ε from inside; hence let $\tilde{B}_\varepsilon \subset B_\varepsilon$ be a convex C^∞ set (in Appendix D it is shown that B can be taken as a cube) with

$$|B_\varepsilon \backslash \tilde{B}_\varepsilon| \leqslant c\varepsilon^{2-d}$$

Let $\tilde{K}_\varepsilon := \tilde{B}_\varepsilon \times (-\infty, +\infty) \subset K_\varepsilon$. Then, denoting again by $m_\varepsilon^{(7)}$ the restriction of this function to $\tilde{K}_\varepsilon$, we have

$$\mathscr{F}^{(1)}(m_\varepsilon^{(7)}; K_\varepsilon) \geqslant \mathscr{F}^{(1)}(m_\varepsilon^{(7)}; \tilde{K}_\varepsilon) - c\varepsilon^{2-d} \tag{4.25}$$

because of the exponential convergence of $m_\varepsilon^{(7)}$ to $\pm m_\beta$ as $r_d \to \pm\infty$.

We denote by $d(r, \partial\tilde{K}_\varepsilon)$ the distance of $r \in \tilde{K}_\varepsilon$ from the boundary $\partial\tilde{K}_\varepsilon$ of $\tilde{K}_\varepsilon$. In Appendix C we prove that there is a smooth function $J^{(2)}(r, r')$: $\tilde{K}_\varepsilon \times \tilde{K}_\varepsilon \to [0, 1]$ with the following properties:

1. $J^{(2)}$ is supported on $\{(r, r'): r \in \tilde{K}_\varepsilon,\ r' \in B_1(r) \cap \tilde{K}_\varepsilon\}$.
2. $J^{(2)}(r, r') = J^{(2)}(r', r)$ on $\tilde{K}_\varepsilon \times \tilde{K}_\varepsilon$.
3. $\int_{\tilde{K}_\varepsilon} dr'\, J^{(2)}(r, r') = 1$ for all $r \in \tilde{K}_\varepsilon$.

4. $J^{(2)}(r, r') = J(|r - r'|)$ for all $r, r' \in \tilde{K}_\varepsilon$ such that $d(r, \partial\tilde{K}_\varepsilon) \geqslant 1$.
5. Set $r = (\xi, r_d)$, $\xi \in \tilde{B}_\varepsilon$, $r_d \in \mathbb{R}$; then for all $\xi \in \tilde{B}_\varepsilon$ and all $r_d, r'_d \in \mathbb{R}$

$$\int_{\tilde{B}_\varepsilon} d\xi' \, J^{(2)}((\xi, r_d), (\xi', r'_d)) = \int_{\tilde{B}_\varepsilon} d\xi' \, J(|(\xi, r_d) - (\xi', r'_d)|) \tag{4.26}$$

We then define for any $m \in L^\infty(\tilde{K}_\varepsilon; [-1, 1])$

$$\mathscr{F}^{(2)}(m; \tilde{K}_\varepsilon) := \int_{\tilde{K}_\varepsilon} dr \, [f(m(r)) - f(m_\beta)] + \tfrac{1}{4} \iint_{\tilde{K}_\varepsilon \times \tilde{K}_\varepsilon} dr \, dr' \, J^{(2)}(r, r')[m(r) - m(r')]^2 \tag{4.27}$$

The same argument used to prove (4.25) shows that

$$\mathscr{F}^{(1)}(m_\varepsilon^{(7)}; \tilde{K}_\varepsilon) \geqslant \mathscr{F}^{(2)}(m_\varepsilon^{(7)}; \tilde{K}_\varepsilon) - c\varepsilon^{2-d} \tag{4.28}$$

By direct inspection one can prove that $\mathscr{F}^{(2)}(m_t; \tilde{K}_\varepsilon)$ is nonincreasing along any orbit m_t solution of (4.19) with K_ε replaced by $\tilde{K}_\varepsilon$ and $J^{(1)}(r, r')$ by $J^{(2)}(r, r')$.

Let $\psi_t \in L^\infty(\mathbb{R}; [-1, 1])$, $t \geqslant 0$, satisfy the equation

$$\frac{d\psi_t(s)}{dt} = -\psi_t(s) + \tanh\{\beta\bar{J} * \psi_t(s)\}, \qquad s \in \mathbb{R} \tag{4.29}$$

with $\bar{J}$ as in (2.18). Then by statement 5 the function $m_t \in L^\infty(\tilde{K}_\varepsilon; [-1, 1])$, $t \geqslant 0$, defined by $m_t(r) := \psi_t(r_d)$ solves (4.19) with $\tilde{K}_\varepsilon$ and $J^{(2)}$. As a consequence, $m^*(r) = \bar{m}(r_d)$, $r \in \tilde{K}_\varepsilon$, $\bar{m}$ as in Theorem 2.6, is a stationary solution of this new version of (4.19).

Let $m_t \in L^\infty(\tilde{K}_\varepsilon; [-1, 1])$ solve this new version of (4.19) with initial condition $m_0 = m_\varepsilon^{(7)}$. Then, by arguments completely similar to the previous ones which led to $m_\varepsilon^{(7)}$, we conclude that $m_t \to m^*$ as $t \to +\infty$. Thus

$$\mathscr{F}^{(2)}(m_\varepsilon^{(7)}; \tilde{K}_\varepsilon) \geqslant \mathscr{F}^{(2)}(m^*; \tilde{K}_\varepsilon) \tag{4.30}$$

Finally, there is a constant $c > 0$ so that, denoting by $|B|$ the area of the original basis B of R,

$$\mathscr{F}^{(2)}(m^*; \tilde{K}_\varepsilon) = \varepsilon^{1-d} s_\beta \, |B| - c\varepsilon^{2-d} \tag{4.31}$$

By (4.24), (4.28), (4.30), and (4.31) we have

$$\mathscr{F}(m_\varepsilon; R_\varepsilon) \geqslant \mathscr{F}^{(2)}(m^*; \tilde{K}_\varepsilon) - c\varepsilon^{2-d} - c\zeta\, |B_\varepsilon| - c \sum_{\mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)})} |S(\mathscr{D})|$$

$$\geqslant \varepsilon^{1-d}(s_\beta - c\zeta)\, |B| - c\varepsilon^{2-d} - c \sum_{\mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)})} |S(\mathscr{D})| \tag{4.32}$$

where $\mathscr{L}^*$ is referred to the whole R_ε. We first consider the sum relative to R_ε^+. By the definition of the minimal section, if $\mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)})$,

$$|S(\mathscr{D})|\, n^* \leqslant |\mathscr{D}| \tag{4.33}$$

with n^* as in (4.7). Moreover,

$$\eta_{m_\varepsilon^{(1)},k,\zeta} < 1 \qquad \text{on any} \quad D \subset \mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)})$$

so that in any such D there is a cube $Q \in \mathscr{Q}^{(k)}$ of side 2^{-k} where

$$\frac{1}{|Q|} \int_Q dr\, m_\varepsilon^{(1)}(r) \leqslant m_\beta - \zeta \tag{4.34}$$

Thus,

$$\int_D dr\, |\chi_R(\varepsilon r) - m_\varepsilon^{(1)}(r)| \geqslant \int_Q dr\, |\chi_R(\varepsilon r) - m_\varepsilon^{(1)}(r)|$$

$$= \int_Q dr\, [m_\beta - m_\varepsilon^{(1)}(r)]$$

$$\geqslant |Q|\, \zeta = 2^{-(k+1)d}\, |D|\, \zeta \tag{4.35}$$

because $Q \subset R_\varepsilon^+$, where $\chi_R = m_\beta$. By (2.23), $\mathscr{F}_\varepsilon = \varepsilon^{1-d} F_\varepsilon$; hence, using (4.33) and recalling (4.7) where n^* is defined, and (4.9), we have

$$\varepsilon^{d-1} \sum_{\mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)})} |S(\mathscr{D})|$$

$$\leqslant \varepsilon^d \sum_{\mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)})} \frac{2\, |\mathscr{D}|}{h}$$

$$\leqslant \varepsilon^d \sum_{\mathscr{D} \in \mathscr{L}^*(m_\varepsilon^{(1)})} \sum_{D \subset \mathscr{D}} \frac{2^{(k+2)d}}{h\zeta} \int_D dr\, |\chi_R(\varepsilon r) - m_\varepsilon^{(1)}(r)|$$

$$\leqslant \frac{c}{h\zeta} \int_R dr\, |\chi_R(r) - u_\varepsilon^{(1)}(r)|$$

$$\leqslant \frac{c}{h\zeta} \int_R dr\, |\chi_R(r) - u_\varepsilon(r)| \tag{4.36}$$

where $u_\varepsilon^{(1)}(r) := m_\varepsilon^{(1)}(\varepsilon^{-1}r)$, $r \in \mathscr{T}$. The same bound is obtained when we sum over the clusters in R_ε^-. By (4.32) and (4.36) we then deduce that

$$F_\varepsilon(u_\varepsilon; R) = \varepsilon^{d-1} \mathscr{F}(m_\varepsilon; R_\varepsilon) \geqslant (s_\beta - c\zeta)\, |B| - c\varepsilon - \frac{c}{h\zeta} \int_R dr\, |\chi_R(r) - u_\varepsilon(r)|$$

Then recalling that $u_\varepsilon \to u$ in $L^1(\mathscr{T})$ as $\varepsilon \to 0^+$, we obtain (4.4). The proposition is thus proved. ∎

4.2. Proof of Condition 2 of Γ-Convergence

We only sketch the proof. Let $u \in BV(\mathscr{T}; \{\pm m_\beta\})$. We first suppose that the boundary ∂E of the set $\{u(r) = m_\beta\}$ is a hypersurface of class $\mathscr{C}^1$. Let $m(r) := u(\varepsilon r)$, $r \in \mathscr{T}_\varepsilon$, so that $m \in L^\infty(\mathscr{T}_\varepsilon; \{\pm m_\beta\})$. Given $0 < \delta < 1$ and $\varepsilon > 0$, we define $u_\varepsilon \in L^\infty(\mathscr{T}_\varepsilon; [-m_\beta, m_\beta])$ as follows. Let $d(r)$ be the signed distance function from $\varepsilon^{-1}\partial E$ positive inside $\varepsilon^{-1}E$. We then set

$$m_\varepsilon^*(r) := \begin{cases} \bar{m}(d(r)) & \text{if } |d(r)| \leqslant \varepsilon^{-\delta} \\ m_\beta & \text{if } d(r) > \varepsilon^{-\delta} \\ -m_\beta & \text{if } d(r) < -\varepsilon^{-\delta} \end{cases}$$

where $\bar{m}(s)$ is the instanton of Theorem 2.6 which converges to $\pm m_\beta$ as $s \to \pm\infty$ exponentially fast. We then have, using the coarea formula (see 3.4.4 in ref. 17)

$$\begin{aligned} \mathscr{F}_\varepsilon(m_\varepsilon^*) \leqslant{}& c' e^{-c\varepsilon^{-\delta}} + \int_{-\varepsilon^{-\delta}}^{\varepsilon^{-\delta}} dt\, [f(\bar{m}(t)) - f(m_\beta)]\, \lambda(\{d(x) = t\}) \\ &+ \tfrac{1}{4} \int_{-\varepsilon^{-\delta}}^{\varepsilon^{-\delta}} dt \int_{-\varepsilon^{-\delta}}^{\varepsilon^{-\delta}} ds\, [\bar{m}(t) - \bar{m}(s)]^2 \int_{\{d(x) = t\}} d\lambda(x) \\ &\times \int_{\{d(y) = s\}} d\lambda(y)\, J(|(x, t) - (y, s)|) \end{aligned}$$

where $d\lambda$ denotes the $(d-1)$-dimensional surface measure. Now for any (x, t) as above and any s

$$\lim_{\varepsilon \to 0^+} \int_{\{d(y) = s\}} d\lambda(y)\, J(|(x, t) - (y, s)|) = \bar{J}(|t - s|)$$

[see (2.18)] and

$$\lim_{\varepsilon \to 0^+} \varepsilon^{d-1} \int_{\{d(x) = t\}} d\lambda(x) = |\partial E|$$

Hence

$$\limsup_{\varepsilon \to 0^+} \varepsilon^{d-1} \mathscr{F}_\varepsilon(m_\varepsilon^*) \leqslant \bar{\mathscr{F}}(\bar{m})\ |\partial E| = s_\beta\ |\partial E|$$

Setting $u_\varepsilon^*(r) := m_\varepsilon^*(\varepsilon^{-1} r)$, $r \in \mathscr{T}$, we then have that $F_\varepsilon(u_\varepsilon^*) \to F(u)$ as $\varepsilon \to 0^+$. This proves condition 2 in the $\mathscr{C}^1$ case. To prove the general case we use a density argument. In Theorem 1.24 of ref. 20 it is proved that every set with bounded perimeter can be approximated in L^1 and in perimeter by a sequence of sets with $\mathscr{C}^\infty$ boundaries. Thus, given any $u \in BV(\mathscr{T}; \{\pm m_\beta\})$, by a diagonalization procedure we then construct a sequence $\{u_\varepsilon\}$ that satisfies condition 2. The Γ-convergence of $\{F_\varepsilon\}$ to F is thus proved. ∎

APPENDIX A

In what follows we denote by $C_\gamma(r)$ the atom of $\mathscr{Q}_\gamma$ that contains r ($\mathscr{Q}_\gamma$ is defined at the beginning of Section 3).

Lemma A1. There is a positive constant c_1 such that for any spin configuration σ_γ, $r \in \mathbb{R}^d$, and $R > 0$ we have

$$\left| \fint_{B_R(r)} dr'\, \sigma_\gamma(r') - \fint_{B_R(r)} dr'\, \pi_\gamma \sigma_\gamma(r') \right| \leqslant c_1 R^{-1} \gamma^\delta$$

Proof. By definition [see (3.4)],

$$\fint_{B_R(r)} dr'\, \pi_\gamma \sigma_\gamma(r') = \fint_{B_R(r)} dr' \frac{1}{|C_\gamma(r')|} \int_{C_\gamma(r')} dr''\, \sigma_\gamma(r'')$$

Then

$$\begin{aligned}
&\fint_{B_R(r)} dr'\, \pi_\gamma \sigma_\gamma(r') - \fint_{B_R(r)} dr'\, \sigma_\gamma(r') \\
&\quad = \fint_{B_R(r)} dr' \frac{1}{|C_\gamma(r')|} \int_{C_\gamma(r') \setminus (C_\gamma(r') \cap B_R(r))} dr''\, \sigma_\gamma(r'') \\
&\quad\quad - \fint_{B_R(r)} dr' \frac{1}{|C_\gamma(r') \cap B_R(r)|} \\
&\quad\quad \times \int_{C_\gamma(r') \cap B_R(r)} dr''\, \sigma_\gamma(r'') \left[1 - \frac{|C_\gamma(r') \cap B_R(r)|}{|C_\gamma(r')|} \right] =: \mathrm{I} + \mathrm{II}
\end{aligned}$$

Since $C_\gamma(r') \cap B_R(r) = C_\gamma(r')$ for $|r' - r| < R - 2\gamma^\delta$ [recall (3.3)], there is a positive constant c_1 such that $\max(\mathrm{I}, \mathrm{II}) \leqslant c_1 R^{-1} \gamma^\delta$. ∎

Lemma A2. There is a positive constant c_2 such that for any spin configuration σ_γ

$$|H_\varepsilon(\sigma_\gamma) - H_\varepsilon(\pi_\gamma \sigma_\gamma)| \leqslant c_2 \gamma^\delta \varepsilon^{-d} \tag{A.1}$$

Proof. Recalling definition (2.3), $H_\varepsilon(\cdot) = H(\cdot; \mathscr{T}_\varepsilon)$, and (3.4), we have

$$\begin{aligned}
&H_\varepsilon(\pi_\gamma \sigma_\gamma) - H_\varepsilon(\sigma_\gamma) \\
&\quad = \frac{1}{2} \int_{\mathscr{T}_\varepsilon} dr_1 \int_{\mathscr{T}_\varepsilon} dr_2 \, J(|r_1 - r_2|) \, \sigma_\gamma(r_1) \, \sigma_\gamma(r_2) \\
&\qquad - \frac{\gamma^{-2\delta d}}{2} \int_{\mathscr{T}_\varepsilon} dr \int_{\mathscr{T}_\varepsilon} dr' \, J(|r - r'|) \int_{C_\gamma(r)} dr_1 \int_{C_\gamma(r')} dr_2 \, \sigma_\gamma(r_1) \, \sigma_\gamma(r_2) \\
&\quad = -\frac{1}{2} \int_{\mathrm{T}_\varepsilon} dr_1 \int_{\mathscr{T}_\varepsilon} dr_2 \, \sigma_\gamma(r_1) \, \sigma_\gamma(r_2) \Bigg[\gamma^{-2\delta d} \int_{C_\gamma(r_1)} dr \\
&\qquad \times \int_{C_\gamma(r_2)} dr' \, J(|r - r'|) - J(|r_1 - r_2|) \Bigg]
\end{aligned}$$

By the regularity of J, there is a positive constant c' so that

$$\sup_{\substack{r \in C_\gamma(r_1) \\ r' \in C_\gamma(r_2)}} |J(|r - r'|) - J(|r_1 - r_2|)| \leqslant c' \mathbf{1}_{\{|r_1 - r_2| \leqslant 2\}} \, \|\nabla J\|_\infty \, \gamma^\delta$$

where $\|\nabla J\|_\infty$ is the sup norm of ∇J and $\mathbf{1}_A$ the characteristic function of A. We then get

$$|H_\varepsilon(\sigma_\gamma) - H_\varepsilon(\pi_\gamma \sigma_\gamma)| \leqslant c_2 \gamma^\delta \varepsilon^{-d} \quad \blacksquare$$

Lemma A3. There is a positive constant c_3 such that for any coarse-grained configuration s_γ

$$|\log W(s_\gamma) - \gamma^{-d} \mathscr{I}_\varepsilon(s_\gamma)| \leqslant c_3 (\gamma \varepsilon)^{-d} \gamma^{2(1-\delta)d} \log(\gamma^{-1})$$

Proof. Let C_γ be an atom of $\mathscr{Q}_\gamma$ and $r \in C_\gamma$. Then, by definition,

$$s_\gamma(r) = \frac{1}{|C_\gamma(r)|} \int_{C_\gamma(r)} dr' \, \sigma_\gamma(r')$$

We call $\lambda_1 \cdots \lambda_N$ the values attained by $\sigma_\gamma(r')$ when r' varies in $C_\gamma(r)$. Thus $\lambda_i \in \{\pm 1\}$ and N goes like $\gamma^{(\delta - 1)d}$. The number of sequences $\{\lambda_i\}$ that give rise to the same value $m \in [-1, 1]$ of $s_\gamma(r)$ is

$$\binom{N}{K(m)}, \qquad \text{where} \quad K(m) := \frac{1 - m}{2} N \in \mathbb{N}$$

We use Stirling's formula to estimate this quantity (see, for instance, ref. 3 and references therein), and we get, recalling the definition of $i(m)$ [see (2.21)],

$$\left|\frac{1}{N}\log\binom{N}{K(m)} - i(m)\right| \leqslant \frac{\log N}{2N} - \frac{1}{N}\log\frac{(1-m)(1+m)}{4} + \frac{4}{N^2(1-m)(1+m)} \leqslant c_3' \frac{\log N}{N} \tag{A.2}$$

for $m = 0, \pm 2/N, \ldots, \pm(N-2)/N$ and for a suitable positive constant c_2'. The left-hand side in (A.2) is 0 if $m = \pm 1$.

The weight $W(s_\gamma)$ of the configuration s_γ is the product of the weights over all the cubes C_γ in $\mathscr{T}_\varepsilon$. We call C_i the generic one and m_i the value of $s_\gamma(r)$ when $r \in C_i$. We then have

$$\left|\sum_{C_i} \log\binom{N}{K(m_i)} - \gamma^{-d}\int_{\mathscr{T}_\varepsilon} dr\, i(m(r))\right| \leqslant c_3(\gamma\varepsilon)^{-d}\gamma^{d(1-\delta)}\log(\gamma^{-1})$$

because the number of terms in the sum is of order $\varepsilon^{-d}\gamma^{-\delta d}$. ∎

Lemma A4. Let N_γ be the total number of distinct coarse-grained configurations s_γ; then there is a positive constant c_4 so that

$$\log N_\gamma \leqslant c_4(\gamma\varepsilon)^{-d}\gamma^{(1-\delta)d}\log(\gamma^{-1})$$

Proof. A configuration s_γ is defined by giving the value of the magnetization in each cube C_γ paving $\mathscr{T}_\varepsilon$. Since the number of possible values of the magnetization is bounded by $2\gamma^{(\delta-1)d}$ and the number of cubes C_γ in $\mathscr{T}_\varepsilon$ is bounded by $\varepsilon^{-d}\gamma^{-\delta d}$, we immediately conclude the proof of the lemma. ∎

APPENDIX B

In this appendix we prove some statements used in the proof of Proposition 4.2. It is convenient to formulate the problem in the following way. We consider three nonempty, disjoint, bounded regions of $\mathbb{R}^d$, Δ, Γ, and Λ. Each one is the union of cubes of the partition $\mathscr{Q}^{(-1)}$. We suppose that $\mathscr{R} := \Delta \cup \Gamma \cup \Lambda$ is connected and that $d(\Delta, \Lambda)$, the Euclidean distance between the two sets, is not smaller than 2.

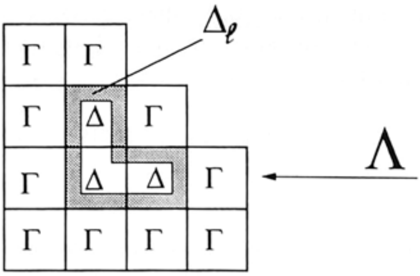

Fig. 4. An example of the regions Δ, Γ, and Λ of Appendix B. Δ is the union of the three central cubes, Γ of the ten cubes surrounding Δ, and Λ is the complement. The shaded region represents Δ_l.

The example we have in mind is with Δ a connected set, and Γ the collection of all the cubes in $\mathscr{R}$ outside Δ that have distance 0 from Δ; Λ is then separated from Δ at least by a cube of $\mathscr{D}^{(-1)}$, whose side has length 2 (see Fig. 4).

We consider the functional $\mathscr{F}(m; \mathscr{R})$ on $L^\infty(\mathscr{R}; [-m_\beta, m_\beta])$ defined in the usual way by

$$\mathscr{F}(m; \mathscr{R}) := \int_{\mathscr{R}} dr\, [f(m(r)) - f(m_\beta)] + \tfrac{1}{4} \iint_{\mathscr{R} \times \mathscr{R}} dr\, dr'\, J(|r-r'|)[m(r) - m(r')]^2 \tag{B.1}$$

Given $l \in (0, 1)$, we set

$$\Delta_l := \{ r \in \Delta \colon d(r, \Gamma) \leqslant l \}; \qquad \Gamma_l := \{ r \in \Gamma \colon d(r, \Delta) \leqslant 1 - l \} \tag{B.2}$$

and finally, for any $r \in \mathscr{R}$, any measurable set $C \subset \mathscr{R}$, and $m \in L^\infty(\mathscr{R}; [-m_\beta, m_\beta])$, we define

$$h(r; m; C) := \frac{\int_C dr'\, J(|r-r'|)\, m(r')}{\int_C dr'\, J(|r-r'|)} \tag{B.3}$$

setting $h(r; m; C) := 1$ if the denominator in (B.3) is 0.

We will consider two cases: $C = \Gamma \cup \Lambda$ and $r \in \Delta_l \cup \Gamma_l$, and $C = (\Gamma \backslash \Gamma_l) \cup \Lambda$ and $r \in \Gamma_l$. In both cases $h(r; m; C)$ does not depend on the values of m in Λ and, moreover, the denominator in (B.3) is uniformly positive because, by Definition 2.1c, $\sup\{ s > 0 \colon J(s) > 0 \} = 1$. This remark is used to prove the following lemma:

Lemma B1. For any $l \in (0, 1)$ there exists a sequence $\{c_k\}_{k \in \mathbb{N}}$ of positive numbers which converges to 0 as $k \to +\infty$ with the following property. If $\zeta > 0$ and for some $k \in \mathbb{N}$

$$\pi^{(k)} m(r) \geqslant m_\beta - \zeta \qquad \text{for all} \quad r \in \Gamma \tag{B.4}$$

then

$$h(r; m; \Gamma \cup \Lambda) \geqslant m_\beta - (\zeta + c_k) \qquad \text{for all} \quad r \in \Delta_l \cup \Gamma_l \tag{B.5}$$

$$h(r; m; (\Gamma \backslash \Gamma_l) \cup \Lambda) \geqslant m_\beta - (\zeta + c_k) \qquad \text{for all} \quad r \in \Gamma_l \tag{B.6}$$

Proof. Let $C = \Gamma \cup \Lambda$ and

$$c_k := \sup_{r \in \Delta_l \cup \Gamma_l} \sup_{m \in L^\infty(\mathcal{R}; [-m_\beta, m_\beta])} |h(r; m; C) - h(r; \pi^{(k)} m; C)| \tag{B.7}$$

Then $c_k \to 0$ as $k \to +\infty$ and if $\pi^{(k)} m(r) \geqslant m_\beta - \zeta$ for $r \in \Gamma$, then

$$h(r; m; C) \geqslant m_\beta - \zeta - c_k \qquad \text{for all} \quad r \in \Delta_l \cup \Gamma_l$$

which proves (B.5). The proof of (B.6) is similar.

Remark. We will actually take for c_k the maximum over all possible choices of Δ and Γ (which is finite because the right-hand side of (B.7) takes only finitely many values).

For any $m \in L^\infty(\mathcal{R}; [-m_\beta, m_\beta])$, $l \in (0, 1)$, and $\theta \in (0, m_\beta)$ we define

$$m_{l,\theta}(r) := \begin{cases} m(r) & \text{if} \quad r \in (\Gamma \backslash \Gamma_l) \cup \Lambda \\ |m(r)| & \text{if} \quad r \in \Delta \cup \Gamma_l \text{ and } |m(r)| \geqslant m_\beta - \theta \\ m_\beta - \theta & \text{if} \quad r \in \Delta \cup \Gamma_l \text{ and } |m(r)| < m_\beta - \theta \end{cases} \tag{B.8}$$

We then have the following result.

Theorem B2. Let $l \in (0, 1)$, $\zeta > 0$, and $k \in \mathbb{N}$ be such that $\zeta + c_k =: \theta < m_\beta$, with c_k defined in (B.7). Then for all m that satisfy (B.4) we have

$$\mathcal{F}(m; \mathcal{R}) \geqslant \mathcal{F}(m_{l,\theta}; \mathcal{R}) \tag{B.9}$$

Proof. Since $m_{l,\theta} = m$ in $(\Gamma \backslash \Gamma_l) \cup \Lambda$ and the support of J is contained in the unit ball, there is no interaction between Λ and $\Delta \cup \Gamma_l$, where m and $m_{l,\theta}$ are different. For this reason the values of m in Λ will not play any role in the following proof.

We observe that under the replacement $m \to m_{l,\theta}$ the first term in the expression of $\mathscr{F}(\cdot; \mathscr{R})$ does not increase, so it is sufficient to prove that the second term has the same property. Given two bounded measurable sets A and B in $\mathbb{R}^d$, we write

$$E(A, B) = \frac{1}{4} \int_{r \in A} \int_{r' \in B} dr\, dr'\, J(|r - r'|) \times \{[m_{l,\theta}(r) - m_{l,\theta}(r')]^2 - [m(r) - m(r')]^2\} \qquad \text{(B.10)}$$

and split the second term in (B.1) into a sum of terms $E(A_i, B_i)$. We will show that each of them is nonpositive and this will prove the theorem. In the proof we will use the two following statements, whose simple proof is omitted:

For any $s \in [-m_\beta, m_\beta]$ and $C \subset \mathscr{R}$ define

$$G(s, C) := \int_C dr'\, J(|r - r'|)[s - m(r')]^2$$

Then:

1. $G(s, C) \geqslant G(t, C)$ for any $s \leqslant t \leqslant h(r; m; C)$.
2. If $h(r; m; C) > 0$, then $G(s, C) \geqslant G(|s|, C)$ for any $s \in [-m_\beta, m_\beta]$.

We next examine separately the various terms $E(A_i, B_i)$:

(T1) $A = B = (\Gamma \backslash \Gamma_l) \cup \Lambda$. Then $E(A, B) = 0$ because $m = m_{l,\theta}$.

(T2) $A = \Gamma_l \cap \{|m(r)| < m_\beta - \theta\}$ and $B = (\Gamma \backslash \Gamma_l) \cup \Lambda$. We apply statement 1 for any $r \in A$ with $s := m(r)$ and $t := m_{l,\theta}(r) = m_\beta - \theta$. In fact, $s \leqslant t \leqslant h(r; m; B)$; the last inequality follows from (B.6) recalling that $\theta = \zeta + c_k$ [see (B.6)]. We then have $G(m(r), B) \geqslant G(m_{l,\theta}(r), B)$, which, integrated over $r \in A$, yields $E(A, B) \leqslant 0$.

(T3) $A = \Gamma_l \cap \{|m(r)| \geqslant m_\beta - \theta\}$ and $B = (\Gamma \backslash \Gamma_l) \cup \Lambda$. For any $r \in A$ we have $h(r; m; B) \geqslant m_\beta - \theta > 0$. Since $m_{l,\theta} = |m|$ on A, we get by statement 2 that $G(m(r), B) \geqslant G(m_{l,\theta}(r), B)$. Integrating this inequality over $r \in A$, we then find that $E(A, B) \leqslant 0$.

(T4) $A = \Delta_l \cap \{|m(r)| < m_\beta - \theta\}$ and $B = \Gamma \cup \Lambda$. We write

$$\begin{aligned} E(A, B) &= \frac{1}{4} \int_A dr \int_B dr'\, J(|r - r'|) \\ &\quad \times \{([m_{l,\theta}(r) - m(r')]^2 - [m(r) - m(r')]^2) \\ &\quad + ([m_{l,\theta}(r) - m_{l,\theta}(r')]^2 - [m_{l,\theta}(r) - m(r')]^2)\} \\ &=: \mathrm{I}(A, B) + \mathrm{II}(A, B) \end{aligned} \qquad \text{(B.11)}$$

For any $r \in A$ we have $h(r; m; B) \geqslant m_\beta - \theta$ by (B.5) and $m_\beta - \theta = m_{l,\theta}(r) =: t \geqslant s := m(r)$. Hence reasoning as in (T2), we get $\mathrm{I}(A, B) \leqslant 0$.

To prove that $\mathrm{II}(A, B) \leqslant 0$ we introduce the sets

$$B_1 := \{r' \in \Gamma_l \colon |m(r')| < m_\beta - \theta\}$$

$$B_2 := \{r' \in \Gamma_l \colon |m(r')| \geqslant m_\beta - \theta\}$$

$$B_3 := \Gamma \backslash \Gamma_l$$

and we split $\mathrm{II}(A, B) = \mathrm{II}(A, B_1) + \mathrm{II}(A, B_2) + \mathrm{II}(A, B_3)$. Then $m_{l,\theta}(r') = m(r')$ for any $r' \in B_3$ so that $\mathrm{II}(A, B_3) = 0$. Moreover, $m_{l,\theta}(r') = m_\beta - \theta$ for $r' \in B_1$ and $m_{l,\theta}(r) \geqslant m_\beta - \theta$ for $r \in \Delta_l$, so that $|m_{l,\theta}(r) - m_{l,\theta}(r')| \leqslant |m_{l,\theta}(r) - m(r')|$ for $r \in A$ and $r' \in B_1$. Therefore $\mathrm{II}(A, B_1) \leqslant 0$. Finally we have $m_{l,\theta}(r') = |m(r')|$ for $r' \in B_2$. Hence

$$\mathrm{II}(A, B_2) = \tfrac{1}{4} \int_{r' \in B_2} dr' \int_{r \in A} dr\, J(|r - r'|) \\ \times \{[|m(r')| - m_{l,\theta}(r)]^2 - [m(r') - m_{l,\theta}(r)]^2\} \leqslant 0$$

by statement 2 because $h(r'; m_{l,\theta}; A) \geqslant m_\beta - \theta > 0$ for any $r' \in B_2$ as $m_{l,\theta} \geqslant m_\beta - \theta$.

(T5) $A = \Delta_l \cap \{|m(r)| \geqslant m_\beta - \theta\}$ and $B = \Gamma \cup A$. We use (B.11) with the new A. The first term on the right-hand side is nonnegative by statement 2: in fact $h(r; m; B) > 0$ for $r \in A$ by (B.5) and $m_{l,\theta} = |m|$ on A by (B.8). For the second term we use the same argument as in the last part of the proof of (T4). Thus $E(A, B) \leqslant 0$.

(T6) $A = B = \Gamma_l$, $A = B = \Delta$ and $A = \Delta \backslash \Delta_l$, $B = \Gamma_l$. In all these cases $|m_{l,\theta}(r) - m_{l,\theta}(r')| \leqslant |m(r) - m(r')|$ for any $r \in A$, $r' \in B$. Hence $E(A, B) \leqslant 0$.

The theorem is proved. ∎

APPENDIX C

In this appendix we prove the properties of the instanton used in the proof of Proposition 4.2. Recall that K_ε is a cylinder in $\mathbb{R}^d$ with basis $B_\varepsilon = \varepsilon^{-1} B$ and that we have supposed for simplicity that K_ε is $\mathscr{Q}^{(0)}$-measurable. The case when K_ε is a torus has been considered in ref. 24. We will first prove the existence of $m_\varepsilon^{(7)}$ (see the proof of Proposition 4.2), hereafter denoted for simplicity by $\tilde{m}$, namely of an instanton solution of (4.19). We will follow ref. 14, where the $d = 1$ case is considered. We use the following basic and elementary properties of the evolution (4.19), where m_t, $t \geqslant 0$, denotes a solution of (4.19) with $m_0 \in L^\infty(K_\varepsilon; [-1, 1])$:

1. Let $\psi_t := m_t - e^{-t} m_0$. Then $|\nabla \psi_t|$ is uniformly bounded with respect to r, t, and m_0. As a consequence there exists a sequence $t_n \to +\infty$ such that m_{t_n} converges uniformly on the compact subsets of K_ε, as $n \to +\infty$.
2. The functional $\mathscr{F}^{(1)}(\cdot; K_\varepsilon)$ defined in (4.16) is lower semicontinuous on $L^\infty(K_\varepsilon; [-1, 1])$ in the weak $L^2_{\mathrm{loc}}(K_\varepsilon)$-topology, and $\mathscr{F}^{(1)}(m; K_\varepsilon) < +\infty$ if and only if there are $\sigma_\pm \in \{\pm 1\}$ such that $m - \chi_\sigma \in L^2(K_\varepsilon, dr)$, where $\chi_\sigma(r) := m_\beta \sigma_+$ [respectively $\chi_\sigma(r) := m_\beta \sigma_-$] if $r_d \geqslant 0$ (respectively $r_d < 0$).
3. If $\mathscr{F}^{(1)}(m_0; K_\varepsilon) < +\infty$, then $d\mathscr{F}^{(1)}(m_t; K_\varepsilon)/dt \leqslant 0$ for all $t \geqslant 0$.
4. If $\mathscr{F}^{(1)}(m_0; K_\varepsilon) < +\infty$, then there are $\hat{m} \in \mathscr{C}(K_\varepsilon; [-m_\beta, m_\beta])$ and a sequence $t_n \to +\infty$ such that $m_{t_n} \to \hat{m}$ uniformly on the compact subsets of K_ε as $n \to +\infty$, and $\hat{m}$ is a stationary solution of (4.19) in the whole K_ε.

We omit the proof of the above properties, which is very similar to that in ref. 14 for $d = 1$.

Proposition C1. There is $\tilde{m} \in \mathscr{C}^\infty(K_\varepsilon; [-m_\beta, m_\beta])$ which is a stationary solution of (4.19). Moreover, $\tilde{m}$ is an antisymmetric and strictly increasing function of r_d.

Proof. Following ref. 14, we set for $r \in K_\varepsilon$

$$m_0(r) := \begin{cases} -m_\beta & \text{for} \quad r_d \leqslant -1 \\ m_\beta & \text{for} \quad r_d \geqslant 1 \\ m_\beta r_d & \text{otherwise} \end{cases} \tag{C.1}$$

Let m_t solve (4.19) with initial datum (C.1). Then m_t is nondecreasing and antisymmetric for any $t \geqslant 0$. Since $\mathscr{F}^{(1)}(m_0; K_\varepsilon) < +\infty$, the limit $\hat{m}$ considered in property 4 solves (C.2) below, is nondecreasing, antisymmetric, and such that $\mathscr{F}^{(1)}(\hat{m}; K_\varepsilon) < +\infty$. This shows that $\tilde{m} := \hat{m}$ is not identically 0.

Recalling that $J^{(1)}$ defined in (4.20) is a $\mathscr{C}^\infty$ function, we then obtain by differentiating the equation

$$\tilde{m}(r) = \tanh\{\beta J^{(1)} * \tilde{m}(r)\}, \qquad r \in K_\varepsilon \tag{C.2}$$

that $\tilde{m} \in \mathscr{C}^\infty(K; [-m_\beta, m_\beta])$. If the derivative $\tilde{m}'$ of $\tilde{m}(r)$ with respect to r_d were 0 at some point r, then it would be 0 in the whole $\{r' : J^{(1)}(r, r') > 0\}$ because $\tilde{m}' \geqslant 0$. By iteration we would then reach a contradiction with the previous statement that $\tilde{m}$ is not identically 0. Hence $\tilde{m}(r)$ is a strictly increasing function of r_d and the proposition is proved. ∎

Proposition C2. There are constants c and c' positive and independent of ε so that the function $\tilde{m}$ satisfies the inequality

$$\tilde{m}(r) \geqslant m_\beta - c' e^{-c r_d}, \qquad r_d \geqslant 0 \tag{C.3}$$

Proof. By Proposition C.1 there is $a > 0$ so that $\tilde{m} \geqslant a$ on $\{r \in K_\varepsilon: r_d \geqslant 1\}$. Using this fact, we will prove that (C.3) holds with a constant c' that depends on a. We will then complete the proof of the proposition by showing that a is uniformly bounded away from zero as $\varepsilon \to 0^+$, hence that c' can be taken independently of ε.

To prove the first statement we introduce the following dynamics. We call $K_\varepsilon^{(i)} := \{r \in K_\varepsilon: r_d \geqslant i\}$, $i = 1, 2$. Then for any $m \in L^\infty(K_\varepsilon^{(1)}; [-1, 1])$ we define $m_t \in L^\infty(K_\varepsilon^{(1)}; [-1, 1])$ as $m_0 := m$ on $K_\varepsilon^{(1)}$, $m_t = m_0$, $t \geqslant 0$, on $K_\varepsilon^{(1)} \backslash K_\varepsilon^{(2)}$. Finally, on $K_\varepsilon^{(2)}$, for $t \geqslant 0$, m_t solves the equation

$$\begin{aligned} \frac{dm_t(r)}{dt} &= -m_t(r) + \tanh\{\beta J^{(1)} * m_t(r)\} \\ J^{(1)} * m_t(r) &= \int_{K_\varepsilon^{(1)}} dr'\, J^{(1)}(r, r')\, m_t(r') \end{aligned} \tag{C.4}$$

$\tilde{m}$ restricted to $K_\varepsilon^{(1)}$ is obviously a stationary solution of (C.4).

Given $u_0 \in L^\infty(K_\varepsilon^{(1)}; [-1, 1])$, we say that the function $u_t \in L^\infty(K_\varepsilon^{(1)}; [-1, 1])$ is a subsolution of (C.4) with initial datum u_0 if $u_t = u_0$, $t \geqslant 0$, on $K_\varepsilon^{(1)} \backslash K_\varepsilon^{(2)}$ and

$$\frac{du_t}{dt} \leqslant -u_t + \tanh\{\beta J^{(1)} * u_t\} \qquad \text{on} \quad K_\varepsilon^{(2)} \text{ for all } t \geqslant 0 \tag{C.5}$$

Since $J^{(1)} \geqslant 0$, one can show that if $u_0 \leqslant m_0$ on $K_\varepsilon^{(1)}$, then $u_t \leqslant m_t$ on $K_\varepsilon^{(1)}$ for all $t \geqslant 0$. A similar statement is proved in ref. 14 for $d = 1$.

We want to construct a subsolution of (C.4). We start by constructing a countable system of functions v_t: $\mathbb{N}_+ \to \mathbb{R}$, $t \geqslant 0$, defined by $v_t(1) := a$, $t \geqslant 0$, and, for $n \in \mathbb{N}$, $n \geqslant 1$,

$$\frac{dv_t(n+1)}{dt} = -v_t(n+1) + \tanh\{\beta v_t(n)\}, \qquad v_0(n+1) = a \tag{C.6}$$

i.e.,

$$v_t(n+1) = e^{-t} a + \int_0^t ds\, e^{-(t-s)} \tanh\{\beta v_s(n)\} \tag{C.7}$$

Since $\beta > 1$, we have $a - \tanh(\beta a) < 0$ (recall that $a < m_\beta$), so that

$$v_t(2) = \tanh(\beta a) + e^{-t}[a - \tanh(\beta a)] \tag{C.8}$$

is an increasing function of t. Define

$$v^*(2) := \lim_{t \to \infty} v_t(2) = \tanh\{\beta a\} \tag{C.9}$$

Similarly, by induction on n, setting

$$v^*(n+1) := \lim_{t \to \infty} v_t(n+1)$$

one can prove that

$$v^*(n+1) = \tanh\{\beta v^*(n)\} \tag{C.10}$$

Calling $T(s) := \tanh\{\beta s\}$ the map from $(0, m_\beta]$ into itself and T^n the nth iterate of T, we have from (C.10)

$$v^*(n+1) = T^n(a)$$

Hence there are constants $c_1 > 0$ and c_2 so that

$$v^*(n) \geqslant m_\beta - c_2 e^{-c_1 n} \tag{C.11}$$

because m_β is a stable point for T attracting any orbit that starts from $(0, m_\beta]$.

To relate the system $v_t(n)$, $n \in \mathbb{N}$, to a subsolution of (C.4) we need preliminarily to prove that $D_t(n+1) := v_t(n+1) - v_t(n) \geqslant 0$ for all $t \geqslant 0$. We have, after Taylor expanding to first order,

$$\begin{aligned} D_t(n+1) &= \int_0^t ds\, e^{-(t-s)}(\tanh\{\beta v_s(n)\} - \tanh\{\beta v_s(n-1)\}) \\ &= \int_0^t ds\, e^{-(t-s)} \psi_s(n)\, D_s(n) \end{aligned} \tag{C.12}$$

where $\psi_s(n)$ is the derivative of the hyperbolic tangent computed at some value which depends on $v_s(n-1)$ and $v_s(n)$. Then $\psi_s(n) > 0$ and, consequently, $D_t(n+1) \geqslant 0$ for all $n \geqslant 1$ and all $t \geqslant 0$, because $D_t(2) \geqslant 0$, as follows from (C.8), recalling that $v_t(1) = a$.

We are going to show that the function $u_t \in L^\infty(K_\varepsilon^{(1)}; [a, m_\beta])$, $t \geqslant 0$, defined by

$$u_t(r) := v_t([r_d]) \tag{3.13}$$

is a subsolution of (C.4). In fact let $r \in K_\varepsilon^{(2)}$ be such that $[r_d] = n+1$; then

$$\begin{aligned} &\frac{du_t(r)}{dt} + u_t(r) - \tanh\{\beta J^{(1)} * u_t(r)\} \\ &\qquad = -\tanh\{\beta J^{(1)} * u_t(r)\} + \tanh\{\beta u_t(n)\} \leqslant 0 \end{aligned} \tag{C.14}$$

because for such values of r one has

$$J^{(1)} * u_t(r) = k_1(r)\, v_t(n+2) + k_2(r)\, v_t(n+1) + k_3(r)\, v_t(n) \geqslant v_t(n) \tag{C.15}$$

for three suitable nonnegative functions k_1, k_2, and k_3. The first equality follows from the fact that $J^{(1)}(r, r') = 0$ if $|r - r'| \geqslant 1$. Moreover, recalling that the r'-integral of $J^{(1)}$ is equal to 1, we get $k_1(r) + k_2(r) + k_3(r) \equiv 1$. The last inequality in (C.15) follows from the fact that $v_t(m) \geqslant v_t(n)$ for $m \geqslant n$, which has already been proved. We have thus shown that u_t satisfies (C.5) and that it is a subsolution of (C.4). Since $\tilde{m} \geqslant u_0$ on $K_\varepsilon^{(1)}$, it then follows that

$$\tilde{m} \geqslant \lim_{t \to +\infty} u_t = u^*, \qquad u^*(r) := v^*([r_d])$$

Then (C.3) follows from (C.11), but, as already observed, the constant c' depends on c_2, which is in turn determined by a. To prove Proposition C.2, we thus need to show that a can be taken independent of ε.

To this end, for any $r \in K_\varepsilon$ with $1 \leqslant r_d \leqslant 2$ we consider a region $\Gamma \subset K_\varepsilon \cap \{1 \leqslant r_d \leqslant 2\}$ containing r which is $\mathscr{D}^{(0)}$-measurable and $*$-connected; see Definitions 2.1a and 4.3. We also suppose that for $\alpha > 0$ as in (C.21) below,

$$\frac{|\delta\Gamma|}{|\Gamma|} \leqslant \alpha \tag{C.16}$$

$|\Gamma|$ is the volume of Γ, hence the number of cubes of $\mathscr{D}^{(0)}$ in Γ; $\delta\Gamma$ is the union of all the cubes in $K_\varepsilon \backslash \Gamma$ at distance not larger than 1 from Γ and $|\delta\Gamma|$ the total number of such cubes. We can find a finite family of such sets Γ so that, modulo translations, for any $\varepsilon > 0$ any r in the strip $\{1 < r_d \leqslant 2\} \cap K_\varepsilon$ is contained in an element of the family. We fix in the sequel a region Γ and call $\Lambda = \Gamma \cup \delta\Gamma$.

Let $u \in L^\infty(\Lambda; \{0, m_\beta\})$, $u(r) := m_\beta$ when $r \in \Gamma$ and $u_0(r) := 0$ when $r \in \delta\Gamma$. We call $u_t \in L^\infty(\Lambda; [0, m_\beta])$, $t \geqslant 0$, the function that solves (C.4) in Γ, with $u_0 = u$ and $u_t = 0$, on $\delta\Gamma$ for all $t \geqslant 0$.

The proposition is then a consequence of the following:

Lemma. (i) There are $u^* \in L^\infty(\Lambda; [0, m_\beta])$ satisfying (C.2) in Γ and a sequence $t_n \to +\infty$ as $n \to +\infty$ such that $u_{t_n} \to u^*$ uniformly on Λ.

(ii) $\tilde{m} \geqslant u^*$ on Λ.

(iii) There is $a > 0$ so that $u^* \geqslant a$ on Γ.

Proof of the Lemma. Statement (i) follows from property 4 (see the beginning of this Appendix), which also applies to $u_t(r)$.

Let $m_t(r)$ be the solution of (4.19) with initial datum (C.1). Then $m_t \geqslant 0$ on $\delta\Gamma$ because $m_t \geqslant 0$ on $\{r_d \geqslant 0\}$, as m_t is antisymmetric and nondecreasing. It then follows that $m_t \geqslant u_t$, $t \geqslant 0$ on Λ.

The proof of (iii) is more delicate. We know from (i) that for $r \in \Gamma$

$$u^*(r) = \tanh\left\{\int_\Lambda dr'\, J^{(1)}(r, r')\, u^*(r')\right\} \tag{C.17}$$

Then there is a constant $c_5 > 0$ such that, for any $r \in \Gamma$,

$$u^*(r) \geqslant c_5 \int_\Gamma dr'\, J^{(1)}(r, r')\, u^*(r') \tag{C.18}$$

Moreover, there are a positive integer n and $a' > 0$ so that for all r and r' in Γ

$$\int_\Gamma dr_1\, J^{(1)}(r, r_1) \cdots \int_\Gamma dr_{n-1}\, J^{(1)}(r_{n-2}, r_{n-1})\, J^{(1)}(r_{n-1}, r') \geqslant a'$$

Thus

$$u^*(r) \geqslant (c_5)^n\, a' \int_\Gamma dr'\, u^*(r') \qquad \text{for all} \quad r \in \Gamma$$

It is therefore sufficient to prove that u^* is not identically 0. To this end we use a Liapunov function for the evolution satisfied by u_t. We set for $u \in L^\infty(\Lambda; [0, m_\beta])$

$$\begin{aligned}\mathscr{F}^{(3)}(u; \Lambda) := \int_\Gamma dr\, [1 - j(r)][f(u(r)) - f(m_\beta)] \\ + \tfrac{1}{4} \int_\Gamma dr \int_\Gamma dr'\, J(|r - r'|)[u(r) - u(r')]^2 \\ + \tfrac{1}{2} \int_\Gamma dr \int_{\delta\Gamma} dr'\, J(|r - r'|)[u(r) - u(r')]^2\end{aligned}$$

One can check that $\mathscr{F}^{(3)}(u_t; \Gamma)$ is a nonincreasing function of t and that

$$\mathscr{F}^{(3)}(u^*; \Lambda) \leqslant \mathscr{F}^{(3)}(u_0; \Lambda) \tag{C.19}$$

We will conclude that $u^* \neq 0$, 0 is here the function identically zero, from the fact that

$$\mathscr{F}^{(3)}(0; \Lambda) > \mathscr{F}^{(3)}(u_0; \Lambda) \tag{C.20}$$

To prove (C.20) we observe that there are positive constants c_3 and c_4 so that

$$\mathscr{F}^{(3)}(0; \Lambda) = \int_\Gamma dr\,[1 - j(r)][f(0) - f(m_\beta)] \geqslant c_3\,|\Gamma|$$

$$\mathscr{F}^{(3)}(u_0; \Gamma) \leqslant \tfrac{1}{2} \int_\Gamma dr \int_{\Lambda \setminus \Gamma} dr'\, J(|r - r'|)\, m_\beta^2 \leqslant c_4\,|\delta\Gamma|$$

Then (C.20) follows from the inequality $c_3\,|\Gamma| > c_4\,|\delta\Gamma|$, which holds if we choose

$$\alpha \equiv \frac{|\delta\Gamma|}{|\Gamma|} = \frac{c_3}{2c_4} \tag{C.21}$$

The lemma and the proposition are therefore proved. ∎

To prove that the instanton $\tilde{m}$ is unique, modulo vertical translations and reflections, and to prove its stability, we follow again the approach used in ref. 14. We thus start from the linearization of (4.19) around $\tilde{m}$. Recalling (4.20), we have

$$\frac{dv}{dt} = Lv, \qquad Lv(r) := -v(r) + \frac{1 - \tilde{m}(r)^2}{1 - j(r)}\, \beta J * v(r) \tag{C.22}$$

L is a self-adjoint operator in $L^2(d\nu; K_\varepsilon)$, where

$$d\nu(r) = dr\, \frac{1 - j(r)}{1 - \tilde{m}(r)^2} \tag{C.23}$$

Its spectrum contains the origin, as the r_d derivative $\tilde{m}'$ of $\tilde{m}$ is a 0-eigenvector,

$$L\tilde{m}' = 0 \tag{C.24}$$

We have proved in Proposition C.2 that $\tilde{m}'$ is in L^2 since it decays exponentially fast.

The spectrum of L is contained in the negative real axis. To prove it, we consider the Perron–Frobenius isomorphism U: $L^2(dv; K_\varepsilon) \to L^2(d\mu; K_\varepsilon)$

$$Uv = \psi \qquad \text{defined by} \quad U^{-1}\psi := \tilde{m}'\psi \tag{C.25}$$

where

$$d\mu(r) := dr\, \tilde{m}'(r)^2 \frac{1 - j(r)}{1 - \tilde{m}(r)^2} \tag{C.26}$$

The image of L under this isomorphism is

$$\mathscr{L} = ULU^{-1} \tag{C.27}$$

which is the self-adjoint Markov generator

$$\begin{aligned} \mathscr{L}\psi(r) &= \int_{K_\varepsilon} dr'\, P(r, r')[\psi(r) - \psi(r')] \\ P(r, r') &:= [1 - \tilde{m}(r)^2] \frac{\tilde{m}'(r')}{\tilde{m}'(r)} \beta J^{(1)}(r, r') \end{aligned} \tag{C.28}$$

The spectrum of $\mathscr{L}$ lies therefore in the negative real axis, the same being true for its isomorphic image L. Denoting by $\langle \cdot, \cdot \rangle_\mu$ the scalar product in $L^2(d\mu; K_\varepsilon)$, we have

$$\langle \psi, L\psi \rangle_\mu = -\tfrac{1}{2} \int_{K_\varepsilon} d\mu(r) \int_{K_\varepsilon} dr'\, P(r, r')[\psi(r) - \psi(r')]^2 \tag{C.29}$$

which shows that 0 is a simple eigenvector, as (C.29) is equal to 0 only if $\psi = c$, a constant; then $U^{-1}\psi = c\tilde{m}'$.

We next show that there is a spectral gap, namely that 0 is an isolated eigenvalue and the rest of the spectrum is at finite distance from 0. We will use Weyl's theorem,[28] and to this end we decompose $L = L_0 + L_1$, where

$$L_0 v := -v + \beta[1 - m_\beta^2]\, J^{(1)} * v \tag{C.30}$$

$$L_1 v := \beta[\tilde{m}^2 - m_\beta^2]\, J^{(1)} * v \tag{C.31}$$

We are going to prove that the spectrum of L_0 is contained in $\{\operatorname{Re} z \leqslant -1 + \beta[1 - m_\beta^2]\}$; recall that $\beta[1 - m_\beta^2] < 1$, because of the definition of m_β. It will suffice to prove that the spectral radius of $1 + L_0$ is not larger than $\beta[1 - m_\beta^2]$. To see this we write

$$\langle [1+L_0]^n v, [1+L_0]^n v\rangle_\mu \leqslant \frac{\|\tilde{m}'\|_\infty^2}{1-m_\beta^2} \langle [1+L_0]^n v, [1+L_0]^n v\rangle_\lambda, \qquad d\lambda(r) = dr[1-j(r)] \tag{C.32}$$

and observe that L_0 is self-adjoint in $L^2(d\lambda; K)$ and

$$(1+L_0)\, v(r) = \int_K dr'\, N(r, r')\, v(r'), \qquad N(r, r') = \beta(1-m_\beta^2)\, J^{(1)}(r, r') \tag{C.33}$$

We then have, calling $c := \|\tilde{m}'\|_\infty^2/(1-m_\beta^2)$,

$$\begin{aligned}\langle [1+L_0]^n v, [1+L_0]^n v\rangle_\lambda &\leqslant c \int_K d\lambda(r) \int_K dr'\, N^{2n}(r, r')\, \tfrac{1}{2}[v(r)^2 + v(r')^2] \\ &\leqslant c[\beta(1-m_\beta^2)]^{2n} \langle v, v\rangle_\lambda \end{aligned} \tag{C.34}$$

Thus $\beta(1-m_\beta^2)$ bounds the spectral radius of $1+L_0$ and this completes the proof of the statements concerning L_0. On the other hand L_1 is a compact operator because it is an integral operator and because, by Proposition C.2, $\tilde{m}(r)$ converges to m_β exponentially fast as $|r_d| \to +\infty$. By applying Weyl's theorem we prove the spectral gap property for L; see ref. 13 for more details.

We have therefore proved that the linearized evolution attracts toward the eigenvector $\tilde{m}'$. Proceeding as in ref. 13, it is possible to prove the local stability of the manifold of instantons, namely the following property. Let $m_0 \in L^\infty(K_\varepsilon; [-1, 1])$. Call $\tilde{m}_a$ the r_d-upward translation by a of $\tilde{m}$. Let μ_a be the measure defined in (C.26) with $\tilde{m}$ replaced by $\tilde{m}_a$. Suppose that there is a so that $m_0 - \tilde{m}_a$ is in $L^2(\mu_a; K_\varepsilon)$ and that its norm is smaller than some suitably fixed value. Then there exists b so that $m(t)$ converges to $\tilde{m}_b$ in $L^2(\mu_a; K_\varepsilon)$.

The remaining part, namely the proof that the instanton $\tilde{m}$ is unique modulo upward translations and reflections and that it is globally stable, is completely similar to the proof of the analogous properties in ref. 14, so that we simply outline the main steps. The key ingredient is a lemma of Fife and McLeod,[19] proved for the Allen–Cahn equation. In the present context it says that given any $m_0 \in L^\infty(K_\varepsilon; [-1, 1])$ such that

$$\liminf_{r_d \to +\infty} m_0(r) > 0, \qquad \limsup_{r_d \to -\infty} m_0(r) < 0 \tag{C.35}$$

there are functions q_t, a_t, and b_t, $t \geqslant 0$, so that for all $t \geqslant 0$

$$\tilde{m}_{a_t} - q_t \leqslant m_t \leqslant \tilde{m}_{b_t} + q_t \tag{C.36}$$

where m_t, $t \geqslant 0$, is the solution of (C.4) with initial datum m_0.

Furthermore q_t, a_t, and b_t converge exponentially fast as $t \to +\infty$ with $q_t \to 0$ as $t \to +\infty$. The proof of (C.36) is just as in ref. 14 and very close to the original one of Fife and McLeod. With the local stability (which is already proven) and (C.36) we can now apply the same argument used in ref. 14 to show that the only stationary solution of (4.19) in K_ε that satisfies (C.35) is translation of the instantons. Using this and exploiting the monotonicity of the functional $\mathscr{F}^{(1)}(\cdot\,; K_\varepsilon)$, one can adapt the Fife–McLeod proof of global stability of the instantons for the Allen–Cahn equation to the present context, just as done in ref. 14 for the $d=1$ case.

Construction of the Kernel $J^{(2)}(r, r')$

Let $S_t(r, \hat{v})$, $t \in \mathbb{R}$, $r \in \tilde{K}_\varepsilon$, $\hat{v} \in \partial B_1(0)$, be the time flow for a point particle in $\tilde{K}_\varepsilon$ with elastic collisions on $\partial \tilde{K}_\varepsilon$. Here $(r, \hat{v})$ denotes the initial position and velocity, t the time, and $S_t(r, \hat{v}) =: (r_t, \hat{v}_t)$ the position and velocity of the particle at time t. Since $\partial \tilde{K}_\varepsilon$ is convex, there are no tangential collisions and we can conclude that, except at the collisions, $S_t(r, \hat{v})$ is smooth and the Jacobian of the transformation $(t, \hat{v}) \to S_t(r, \hat{v})$ is nonzero. Moreover, by the Liouville theorem, the flow S_t preserves the Lebesgue measure on $\tilde{K}_\varepsilon \times \partial B_1(0)$.

We call $\lambda(d\hat{v})$ the surface measure on $\partial B_1(0)$ and

$$p(dt\, d\hat{v}) := J(t)\, t^{d-1}\, dt\, \lambda(d\hat{v})$$

We define $J^{(2)}(r, r')$ by setting, for any $f \in L^\infty(\tilde{K}_\varepsilon)$ (also thought of as a function on $\tilde{K}_\varepsilon \times \partial B_1(0)$ that does not depend on $\hat{v}$) and any $r \in \tilde{K}_\varepsilon$,

$$\int_{\tilde{K}_\varepsilon} dr'\, J^{(2)}(r, r')\, f(r') = \int_{[0,1] \times \partial B_1(0)} p(dt\, d\hat{v})\, f(S_t(r, \hat{v}))$$

Property 1 of $J^{(2)}$ follows from the fact that $t \leqslant 1$ and that the speed is 1; the equality

$$\int_{[0,1] \times \partial B_1(0)} p(dt\, d\hat{v}) = \int_{\mathbb{R}^d} dr'\, J(r, r') = 1$$

proves property 3. Property 4 holds because $S_t(r, \hat{v}) = r + t\hat{v}$ if $d(r, \partial \tilde{K}_\varepsilon) \geqslant 1$; in fact in such a case the point does not collide before $t=1$. Property 5 follows from the fact that the evolution of the last coordinate r_d is as in the free motion, as the walls $\partial \tilde{K}_\varepsilon$ are parallel to the r_d axis.

To prove property 2 we write for $g, f \in L^\infty(\tilde{K}_\varepsilon \times \partial B_1(0))$ which do not depend on $\hat{v}$ but only on r

$$\int_{\tilde{K}_\varepsilon \times \tilde{K}_\varepsilon} dr\, dr'\, J^{(2)}(r, r')\, g(r)\, f(r')$$
$$= \int_0^1 dt\, J(t)\, t^{d-1} \int_{\tilde{K}_\varepsilon} dr \int_{\partial B_1(0)} \lambda(d\hat{v})\, g(r)\, f(S_t(r, \hat{v}))$$
$$= \int_0^1 dt\, J(t)\, t^{d-1} \int_{\tilde{K}_\varepsilon} dr' \int_{\partial B_1(0)} \lambda(d\hat{v}')\, f(r')\, g(S_{-t}(r', \hat{v}'))$$

where we used the Liouville theorem. We now observe that the position coordinate is the same in $S_{-t}(r', \hat{v}')$ and $S_t(r', -\hat{v}')$ and since g does not depend on the velocity, we may write as well $g(S_t(r', -\hat{v}'))$ in the last integral. By the symmetry of the measure $d\lambda$ under the change $\hat{v} \to -\hat{v}$ we then complete the proof of property 2. All the properties of $J^{(2)}(r, r')$ listed in Section 4 have been proved.

APPENDIX D

In this appendix we recall some basic notions of geometric measure theory; we refer to the book of Evans and Gariepy[17] for more details.

We recall that a set $E \subset \mathscr{T}$ has finite perimeter when its characteristic function $\mathbf{1}_E$ belongs to the space $BV(\mathscr{T})$ of the functions on $\mathscr{T}$ of bounded variation. When E has finite perimeter, there exists a set $\partial^* E$ (reduced boundary of E) and a function $v: \partial^* E \to \mathbb{R}^d$, $|v| = 1$ (generalized outer normal to $\partial^* E$), such that for every vector field $\phi \in \mathscr{C}^1(\mathscr{T})$ the following generalized form of Gauss–Green formula holds:

$$\int_E dr \operatorname{div} \phi = -\int_{\partial^* E} d\lambda\, \phi \cdot v \tag{D.1}$$

where $d\lambda$ denotes (a suitable extension of) the $(d-1)$-dimensional surface measure. The set $\partial^* E$ is rectifiable in a measure-theoretic sense; the precise statement may be found in Section 5.7 of ref. 17.

The perimeter $P(E)$ of E is given by the surface measure $\lambda(\partial^* E)$ of its reduced boundary. When $u \in BV(\mathscr{T}; \{\pm m_\beta\})$, we define the perimeter $P(u)$ of u as the perimeter of the set

$$E := \{r \in \mathscr{T} : u(r) = -m_\beta\} \tag{D.2}$$

The following approximation result can be found in ref. 8.

Theorem D1. Let E, $\partial^* E$, and v be as above. Then for any $\zeta > 0$ there exists a set F of class $\mathscr{C}^1$ such that

$$|E \triangle F| < \zeta, \qquad \lambda(\partial^* E \triangle \partial F) < \zeta \tag{D.3}$$

where $\triangle$ denotes the symmetric difference of sets.

The following result has been used in Section 4:

Theorem D2 (Covering theorem). Let $u \in BV(\mathscr{T}; \{\pm m_\beta\})$. Then for any $0 < \zeta < 1$ there is $h > 0$ and there are disjoint parallelepipeds $R_1,..., R_n$ in $\mathbb{R}^d$ with bases $\mathbb{R}^{d-1}$-parallelepipeds $B_1,..., B_n$, respectively, and equal height $2h$, so that

$$\frac{1}{h} \sum_{i=1}^{n} \int_{R_i} dr\, |\chi_{R_i} - u| < \zeta; \qquad \left| \sum_{i=1}^{n} |B_i| - P(u) \right| < \zeta \tag{D.4}$$

where $\chi_{R_i} := m_\beta(\mathbf{1}_{R_i^+} - \mathbf{1}_{R_i^-})$.

It is possible to take all the R_i congruent to the same d-dimensional cube R of size $2h$.

Proof. (Sketch.) Using Theorem D.1, we may restrict ourselves to the case ∂E of class $\mathscr{C}^1$; see (D.2). In what follows we denote by c_i positive constants, possibly depending on d and m_β, but not on ζ and h.

We can find pairwise disjoint open subsets $\Sigma_1,..., \Sigma_m$ of ∂E which cover ∂E up to a set of (surface) measure less than $c_1 \zeta$, and so that each Σ_i is (congruent to) the graph of a real function f_i: $U_i \to \mathbb{R}$ of class $\mathscr{C}^1$, where $U_i \subset \mathbb{R}^{d-1}$ is a bounded open set and f_i satisfies the bound

$$|\nabla f_i| \leqslant c_2 \zeta \qquad \text{on} \quad U_i \tag{D.5}$$

See Figs. 5 and 6.

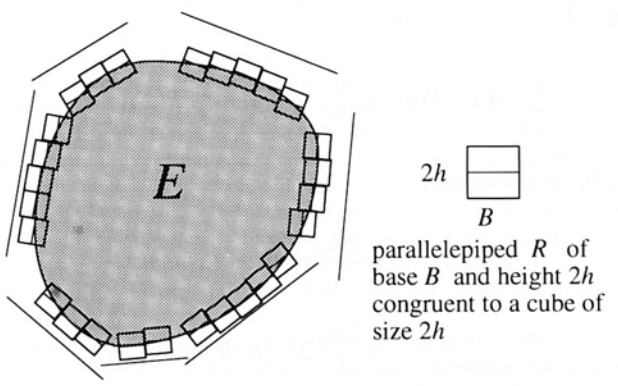

Fig. 5. The shaded region is a set E with finite perimeter. Shown is a covering of parts of ∂E by parallelepipeds as in Theorem D.2. Locally ∂E is the graph of a function; the segments outside E represent symbolically the domains of definition of these functions.

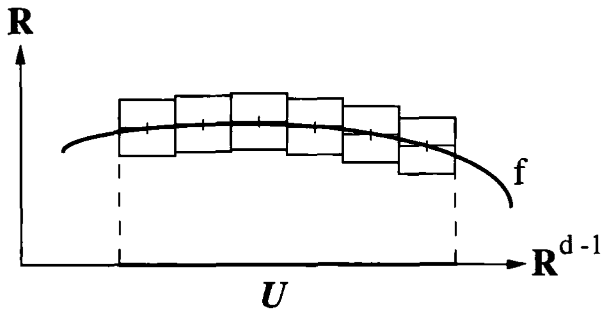

Fig. 6. Blowup of a single piece of the covering of Fig. 5.

We choose now $h > 0$ small enough in such a way that the following two conditions hold:

(i) The number $2\sqrt{d}\,h$ (which is the length of the diagonal of a d-dimensional cube of side $2h$) is less than the distance between Σ_i and Σ_j for any $i \neq j$.

(ii) For every i we cover U_i with pairwise disjoint $(d-1)$-dimensional cubes $B \subset U_i$ of side $2h$ up to a set of measure less than $c_3\zeta/m$.

For any $i = 1,..., m$ and any $(d-1)$-dimensional cube $B \subset U_i$ appearing in (ii) centered at x we construct the d-dimensional cube $R \subset \mathbb{R}^d$ with basis B centered at the point $(x, f_i(x))$ and with height $2h$; see Fig. 6. Denote by $R_1,..., R_n$ (with bases $B_1,..., B_n$, respectively) the collection of all these cubes. We also assume that R_i^- is in the direction of E.

Using (D.5), we then have

$$\left|\sum_{i=1}^{n} |B_i| - P(u)\right| \leqslant \sum_{i=1}^{m} \left(c_3 \frac{\zeta}{m} + c_4 \zeta P(\Sigma_i)\right) \leqslant c_5 \zeta$$

Moreover, by construction,

$$\frac{1}{h}\int_{R_i} dr\, |\chi_{R_i} - u| = \frac{2m_\beta}{h}\left(|R_i^- \cap E| + |R_i^+ \cap E^c|\right)$$

$$\leqslant \frac{2m_\beta}{h} 2c_2\zeta h\, |B_i| \leqslant c_6 \zeta\, |B_i|$$

with E^c the complement of E. Finally, thanks to (D.5) and (i) one can show that the R_i are pairwise disjoint for a suitable choice of the constants c_i. ∎

ACKNOWLEDGMENTS

We are indebted to Pierre Picco for many useful discussions and for helpful suggestions in the proof of Proposition 4.2. This research has been partially supported by CNR (GNFM and GNSM) and by grant CEE CHRX-CT93-0411.

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